

# Perturbation theory for self-gravitating gauge fields I: The odd-parity sector

O. Sarbach\*, M. Heusler\*, and O. Brodbeck†

\**Institute for Theoretical Physics, University of Zurich, CH-8057 Zurich, Switzerland*

†*Max-Planck-Institute for Physics, Werner Heisenberg Institute, D-80805 Munich, Germany*

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A gauge- and coordinate-invariant perturbation theory for self-gravitating non-Abelian gauge fields is developed and used to analyze local uniqueness and linear stability properties of non-Abelian equilibrium configurations. It is shown that all admissible stationary odd-parity excitations of the static and spherically symmetric Einstein-Yang-Mills soliton and black hole solutions have total angular momentum number  $\ell = 1$ , and are characterized by non-vanishing asymptotic flux integrals. Local uniqueness results with respect to non-Abelian perturbations are also established for the Schwarzschild and the Reissner-Nordström solutions, which, in addition, are shown to be linearly stable under dynamical Einstein-Yang-Mills perturbations. Finally, unstable modes with  $\ell = 1$  are also excluded for the static and spherically symmetric non-Abelian solitons and black holes. (PACS numbers: 04.25.Nx, 04.40.-b, 04.70.Bw)

## I. INTRODUCTION

Self-gravitating non-Abelian gauge fields admit a rich spectrum of equilibrium configurations, which is a consequence of the balance between the gravitational attraction and the repulsive nature of the Yang-Mills interaction. In particular, the static and spherically symmetric non-Abelian soliton [1] and black hole solutions [2] owe their existence to the nonlinearities of *both* general relativity and Yang-Mills theory.

On the other hand, the key to the black hole uniqueness theorems [3] lies in the  $\sigma$ -model structure of the Einstein(-Maxwell) equations in the presence of a Killing field [4], [5]. As this property ceases to exist for self-gravitating *non-Abelian* gauge fields [6], the classification of all stationary Einstein-Yang-Mills (EYM) soliton and black hole solutions is necessarily a very difficult task. In particular, the set of global charges (asymptotic flux integrals) does no longer uniquely characterize all possible non-Abelian equilibrium configurations.

Induced by the work of Bartnik and McKinnon (BK) on non-Abelian solitons [1], various new self-gravitating equilibrium configurations have been found during the last decade. Besides the abovementioned static and spherically symmetric black holes with Yang-Mills hair (i.e., with vanishing Yang-Mills charges but different metric structure than the Schwarzschild solution) [2], these include soliton and black hole solutions in Skyrme, Higgs, dilaton and other non-linear field theories coupled to gravity (see [7] for a review and references).

Moreover, numerical [8] and analytical [9] studies have revealed that non-Abelian static black holes are not necessarily spherically symmetric – in fact, they need not even be axisymmetric [10]. In addition, the non-linear nature of the Yang-Mills interaction enables the existence of stationary, non-static black holes with vanishing Komar angular momentum [11]. Also, the usual Lewis-Papapetrou form of the metric does not necessarily describe all stationary and axisymmetric EYM black holes, that is, the *circularity* theorem does not generalize to

space-times containing non-Abelian gauge fields [12].

The above comments suggest that it is not (yet) feasible to completely classify the soliton and black hole solutions of the stationary EYM equations. In this article we pursue, therefore, a more modest aim. That is, we compute the complete spectrum of *stationary EYM perturbations* of the BK solitons and the corresponding black holes with hair. We do so by systematically developing the perturbation theory for self-gravitating non-Abelian gauge fields. Following the tradition, we start with the odd-parity sector, and defer the investigation of even-parity perturbations to a forthcoming publication [13].

The gauge- and coordinate-invariant equations derived in this paper describe perturbations of arbitrary spherically symmetric EYM configurations, where the stationary and the dynamical sector can be treated separately if the background is static. In order to classify the equilibrium solutions close to the BK solitons and the corresponding black holes, it is sufficient to consider *stationary* excitations. As we shall see, these are naturally analyzed in terms of invariant *metric* and Yang-Mills amplitudes.

The main results of this paper concern two local uniqueness theorems, applying to the BK solitons and the corresponding black holes with hair, respectively: We prove that all stationary odd-parity excitations of these static and spherically symmetric configurations are parametrized in terms of infinitesimal asymptotic flux integrals. More precisely, we show that the soliton and black hole excitations found in [11] are the only stationary, asymptotically flat perturbations of the BK solitons and the corresponding black holes with hair. In particular, there exist no admissible regular or black hole perturbations with total angular momentum number  $\ell > 1$ , while for  $\ell = 1$ , the unique soliton and black hole excitations are those with infinitesimal electric charge and/or infinitesimal Komar angular momentum [11]. On the perturbative level, the situation is, therefore, similar to the Abelian case, where the only admissible stationary excitations of the Schwarzschild metric are the Kerr-Newman modes. The above results also establish a local version

of the circularity theorem in the odd-parity sector.

In addition to the classification of neighboring equilibrium configurations, we also discuss some *stability* issues, which require the analysis of *dynamical* perturbations. Unfortunately, the gauge-invariant metric perturbations used in this paper are, in general, not suited to apply spectral analysis, since their evolution is not governed by a standard pulsation operator. In a recent work [14] we have demonstrated how to overcome this problem by using *curvature-based* quantities. A rigorous discussion of dynamical perturbations within the *metric* approach is nevertheless possible for some distinguished cases. These include  $\ell = 1$  EYM perturbations of arbitrary background configurations, and arbitrary EYM perturbations of embedded Abelian configurations.

Hence, further results derived in this paper concern the non-Abelian stability (and local uniqueness) of the Schwarzschild and the Reissner-Nordström (RN) black holes, as well as the stability properties of non-Abelian configurations with respect to  $\ell = 1$  perturbations. In particular, we show that both the Schwarzschild and the RN metric are linearly stable with respect to dynamical non-Abelian perturbations and admit no stationary excitations other than the (embedded) Kerr-Newman modes. In addition, we establish the absence of unstable modes of the pulsation equations governing the  $\ell = 1$  perturbations of the BK solitons and the corresponding black holes with hair. In this context it is worthwhile recalling that unstable Yang-Mills modes with odd-parity do exist for  $\ell = 0$  [15].

The paper is organized as follows: In Section II we briefly review the gauge-invariant approach to odd-parity gravitational perturbations and give a coordinate-invariant derivation of the Regge-Wheeler (RW) equation. In Section III we present the harmonic decomposition of Yang-Mills fields, using a convenient method to parametrize  $\text{su}(2)$ -valued one-forms in terms of isospin harmonics. Taking advantage of some powerful tools developed in Appendix D, the linearized field equations governing arbitrary odd-parity perturbations of spherically symmetric EYM configurations are derived in Section IV.

As first applications, we establish the linear stability and the local uniqueness properties of the Schwarzschild and the RN solutions with respect to non-Abelian perturbations in Sections V and VI, respectively. The local uniqueness theorems for the BK solitons and the corresponding black holes are proven in Section VII. Eventually, in Section VIII, we establish the dynamical stability of these solutions with respect to non-spherical perturbations with  $\ell = 1$ .

A variety of technical issues, such as the expressions for the linearized Ricci tensor, the integral argument excluding admissible solutions of certain RW type equations, some asymptotic expansions, the introduction of isospin harmonics, and the construction of gauge- and coordinate-invariant Yang-Mills amplitudes are discussed in Appendixes A-F.

## II. GRAVITATIONAL PERTURBATIONS

In this section we briefly review the gauge-invariant approach to odd-parity gravitational perturbations [16]. As an application we derive a coordinate-invariant version of the RW equation [17]. We finally recall the arguments establishing the stability of the Schwarzschild metric with respect to vacuum perturbations.

### A. Background expressions

We are analyzing odd-parity perturbations of *spherically symmetric background* configurations. A spherically symmetric spacetime  $(M, g)$  is a warped product of  $\tilde{M} \equiv M/\text{SO}(3)$  and  $S^2$  with metric

$$g = \tilde{g} + R^2 \hat{g}. \quad (1)$$

Here  $\hat{g}$  is the standard metric on  $S^2$ , and  $\tilde{g}$  and  $R$  denote the metric tensor and a real-valued function, respectively, defined on the two-dimensional pseudo-Riemannian orbit space  $\tilde{M}$  with coordinates  $x^a$ , say. Here and in the following lower-case Latin indices ( $a = 0, 1$ ) refer to coordinates on  $(\tilde{M}, \tilde{g})$ , while capital Latin indices ( $A = 2, 3$ ) refer to the coordinates  $\vartheta$  and  $\varphi$  on  $(S^2, \hat{g})$ . The dimensional reduction of the Einstein tensor yields

$$\begin{aligned} G_{ab} &= \frac{1}{R^2} \left( 2R\tilde{\Delta}R + \langle dR, dR \rangle - 1 \right) \tilde{g}_{ab} - \frac{2}{R} \tilde{\nabla}_a \tilde{\nabla}_b R, \\ G_{AB} &= \frac{1}{2} \left( 2R\tilde{\Delta}R - R^2 \tilde{R} \right) \hat{g}_{AB}, \end{aligned} \quad (2)$$

where the off-diagonal components vanish,  $G_{Ab} = 0$ . The operators with a tilde and the inner product  $\langle \cdot, \cdot \rangle$  refer to the two-dimensional pseudo-Riemannian metric  $\tilde{g}$ , and  $\tilde{R}$  denotes the Ricci scalar of  $\tilde{g}$ .

### B. Coordinate-invariant amplitudes

Arbitrary perturbations of spherically symmetric background fields can be expanded in terms of spherical tensor harmonics. For odd-parity perturbations the transverse spherical vector harmonics,  $S_A \equiv (\hat{*}dY)_A$  form a basis of vector fields on  $S^2$ , while the harmonics  $\hat{\nabla}_{\{A} S_{B\}} \equiv \frac{1}{2}(\hat{\nabla}_A S_B + \hat{\nabla}_B S_A)$  are a basis of symmetric tensor fields on  $S^2$ ; see Appendix D for details. (Here  $\hat{*}$  denotes the Hodge dual with respect to the metric  $\hat{g}$ , and the  $Y^{\ell m}$  are the scalar spherical harmonics, where the angular numbers  $\ell$  and  $m$  are suppressed throughout, i.e.,  $Y \equiv Y^{\ell m}$ ,  $S_A \equiv S_A^{\ell m}$ .) The odd-parity perturbations of  $g_{\mu\nu}$  are, therefore, parametrized in terms of a scalar field  $\kappa$  and a one-form  $h = h_a dx^a$ ,

$$\delta g_{ab} = 0, \quad \delta g_{Ab} = h_b S_A, \quad \delta g_{AB} = 2\kappa \hat{\nabla}_{\{A} S_{B\}}, \quad (3)$$

where  $\kappa$  and  $h_a$  depend on the coordinates  $x^b$  only.

A vector field  $X = X^\mu \partial_\mu$  generating an infinitesimal coordinate transformation with odd parity is determined by a function  $f(x^b)$ , where

$$X^a = 0, \quad X^A = f S^A = \frac{f}{R^2} \hat{g}^{AB} S_B. \quad (4)$$

Under coordinate transformations induced by  $X$  the perturbations of a tensor field transform with the Lie derivative of the corresponding background quantity with respect to  $X$ :  $\delta t_{\mu\nu} \rightarrow \delta t_{\mu\nu} + \mathcal{L}_X t_{\mu\nu}$ . Using  $\mathcal{L}_X g_{Ab} = S_A R^2 \hat{\nabla}_b (R^{-2} f)$  and  $\mathcal{L}_X g_{AB} = 2f \hat{\nabla}_{\{A} S_{B\}}$ , the metric perturbations transform according to

$$\kappa \rightarrow \kappa + f, \quad \frac{h_b}{R^2} \rightarrow \frac{h_b}{R^2} + \hat{\nabla}_b \left( \frac{f}{R^2} \right). \quad (5)$$

In a similar way one obtains the transformation laws for the perturbations of the Einstein tensor. Also using the background properties  $G_{Ab} = 0$  and  $2G_B^A = G_D^D \delta_B^A$  one finds

$$\delta G_{Ab} \rightarrow \delta G_{Ab} + G_A^B S_B R^2 \hat{\nabla}_b \left( \frac{f}{R^2} \right), \quad (6)$$

$$\delta G_{AB} \rightarrow \delta G_{AB} + G_D^D \hat{\nabla}_{\{A} S_{B\}} f. \quad (7)$$

One may now use the transformation laws for  $\kappa$  and  $h_b$  to construct the following coordinate-invariant components:

$$\delta G_{ab}^{inv} \equiv \delta G_{ab}, \quad \delta G_{Ab}^{inv} \equiv \delta G_{Ab} - h_b G_A^B S_B, \quad (8)$$

and, for  $\ell \neq 1$ ,

$$\delta G_{AB}^{inv} \equiv \delta G_{AB} - \kappa G_D^D \hat{\nabla}_{\{A} S_{B\}}. \quad (9)$$

We recall that the scalar amplitude  $\kappa$  defined in Eq. (3) is not present for  $\ell = 1$ , since then  $\hat{\nabla}_{\{A} S_{B\}}$  vanishes. However, by virtue of Eq. (7), this also implies that  $\delta G_{AB}$  is already coordinate-invariant. (In fact,  $\delta G_{AB}$  vanishes identically for  $\ell = 1$ , as will be shown below.) Hence, for  $\ell = 1$  one needs only the invariant components defined in Eqs. (8), which do not involve the amplitude  $\kappa$ .

As the  $\delta G_{\mu\nu}^{inv}$  are invariant under coordinate transformations generated by  $X$ , the expressions (8) and (9) will only involve coordinate-invariant combinations of the one-form  $h$  and the scalar  $\kappa$ . In fact, for  $\ell \neq 1$ ,  $\delta G_{\mu\nu}^{inv}$  can be expressed in terms of the manifestly coordinate-invariant one-form  $H$ , defined by

$$H \equiv h - R^2 d \left( \frac{\kappa}{R^2} \right). \quad (10)$$

This definition is again limited to  $\ell \neq 1$ . For  $\ell = 1$ , where  $\kappa$  is absent, we will see that the remaining perturbation  $h$  enters  $\delta G_{\mu\nu}^{inv}$  via the invariant two-form  $d(R^{-2}h)$  only.

### C. Coordinate-invariant Einstein tensor

The computation of the coordinate-invariant components  $\delta G_{\mu\nu}^{inv}$  is considerably simplified by the following observation: In the gauge where the scalar amplitude  $\kappa$  vanishes, henceforth called the off-diagonal gauge (ODG), the perturbation  $h$  coincides with the coordinate-invariant perturbation  $H$  defined in Eq. (10). (It is obvious from Eq. (5) that the ODG always exists and fixes the gauge function  $f$  uniquely.) Hence, for  $\ell > 1$ , the correct invariant tensors are obtained by computing  $\delta G_{\mu\nu}^{inv}$  in the ODG, and by substituting  $H$  for  $h$  in the resulting expressions. For  $\ell = 1$  all perturbations are off-diagonal anyway, and one obtains the correct expressions in terms of the invariant quantity  $d(R^{-2}h)$ .

It is a straightforward task to compute  $\delta G_{\mu\nu}$  in the ODG. Using the formulas (A3), (A4) and (A5) derived in Appendix A, Eqs. (8) and (9) yield the expressions

$$\begin{aligned} \delta G_{Ab}^{inv} |_{ODG} &= \frac{S_A}{R^2} \left\{ \hat{\nabla}^a \left[ R^4 \hat{\nabla}_{[b} (h_a R^{-2}) \right] + \frac{\lambda}{2} h_b \right\}, \\ \delta G_{ab}^{inv} |_{ODG} &= 0, \quad \delta G_{AB}^{inv} |_{ODG} = \hat{\nabla}_{\{A} S_{B\}} \hat{\nabla}^b h_b, \end{aligned} \quad (11)$$

where

$$\lambda \equiv (\ell - 1)(\ell + 2).$$

Here we have used the background property  $2G_B^A = G_D^D \delta_B^A$  and the fact that  $\delta G_{AB}^{inv} = \delta G_{AB}$  in the ODG. Since  $h_b$  coincides with the invariant amplitude  $H_b$  in the ODG, we may replace  $h_b$  by  $H_b$  in the above expressions, which makes them manifestly coordinate-invariant for  $\ell > 1$ . For  $\ell = 1$  the second term in the expression for  $\delta G_{Ab}^{inv}$  vanishes, and  $h_b$  appears only via the coordinate-invariant expression  $\hat{\nabla}_{[b} (h_a R^{-2})$ . We therefore end up with the manifestly coordinate-invariant expressions

$$\delta G_{ab}^{inv} = 0, \quad \delta G_{AB}^{inv} = -d^\dagger H \hat{\nabla}_{\{A} S_{B\}} \quad (12)$$

and

$$\delta G_{Ab}^{inv} dx^b = \frac{S_A}{2R^2} \left\{ d^\dagger [R^4 d(R^{-2}H)] + \lambda H \right\}, \quad (13)$$

which are valid for all values of  $\ell$ , provided that  $H$  is defined according to Eq. (10) for  $\ell > 1$ , and according to  $H \equiv h$  for  $\ell = 1$ . Here  $d^\dagger \equiv \tilde{*}d\tilde{*}$  denotes the co-differential operator for  $p$ -forms on  $(\tilde{M}, \tilde{g})$ , e.g.,  $d^\dagger H = -\hat{\nabla}^a H_a$ ,  $(d^\dagger dH)_b = 2\hat{\nabla}^a \hat{\nabla}_{[b} H_{a]}$ .

The linearized Bianchi identity implies that the Einstein equation for  $\delta G_{AB}^{inv}$  is a consequence of the equation for  $\delta G_{Ab}^{inv}$ . In fact, the first equation is the integrability condition for the second one, as is obvious for vacuum perturbations: Applying the co-differential to  $R^2 \delta G_{Ab}^{inv} = 0$  yields  $d^\dagger H = 0$ , that is,  $\delta G_{AB}^{inv} = 0$ . (For  $\ell = 1$  this integrability condition is void, in agreement with the fact that  $\delta G_{AB}^{inv}$  vanishes identically.)

### D. Local uniqueness and linear stability of the Schwarzschild metric

As an application we consider vacuum perturbations of the Schwarzschild metric. The relevant equation for the odd-parity sector was first derived by Regge and Wheeler [17], and brought in a gauge-invariant form by Gerlach and Sengupta [16]. A gauge-invariant approach which is based on the Hamiltonian formalism was given by Moncrief [18].

The *linear stability* of the Schwarzschild metric follows from the dynamical behavior of vacuum fluctuations. In order to establish the *local uniqueness* property one also has to exclude all stationary perturbations other than the Kerr mode. While the stationary perturbations do not need to be normalizable, they are, however, subject to certain boundary conditions following from asymptotic flatness and regularity requirements. Both stationary and dynamical perturbations must be analyzed separately in the sectors  $\ell > 1$  and  $\ell = 1$ .

The vacuum perturbations with odd parity are obtained from Eq. (13), which yields

$$\frac{1}{R^2} d^\dagger \left[ R^4 d \left( \frac{H}{R^2} \right) \right] + \lambda \frac{H}{R^2} = 0. \quad (14)$$

This equation holds for all values of  $\ell$  and comprises the complete information. The usual way to derive the RW equation from Eq. (14) is to decompose the one-form  $H$  with respect to Schwarzschild coordinates,  $H = H_t dt + H_r dr$ , and to use the integrability condition to eliminate  $H_t$ . This yields an equation for  $H_r$  alone, which is then cast into a wave equation for the function  $(1 - 2M/r)H_r/r$ . This can also be achieved in a coordinate-invariant way as follows: Using the integrability condition  $d^\dagger H = 0$  to introduce the scalar potential  $\Phi$  according to  $H = \tilde{*}d(R\Phi)$ , one may integrate Eq. (14). This yields Eq. (15) below for the potential  $\Phi$  instead of  $\Psi$ .

Here we proceed in a different way, which is also coordinate-invariant. The basic observation is that in two dimensions the field strength two-form assigned to a one-form is equivalent to a scalar field. We therefore introduce the scalar field  $\Psi$  according to

$$\Psi \equiv R^3 \tilde{*}d \left( \frac{H}{R^2} \right),$$

where the factor  $R^3$  turns out to be convenient. Applying the operator  $\tilde{*}d$  on Eq. (14) and using the above definition yields the wave equation

$$\left[ -\tilde{\Delta} + R\tilde{\Delta} \left( \frac{1}{R} \right) + \frac{\lambda}{R^2} \right] \Psi = 0, \quad (15)$$

where the two-dimensional Laplacian of a function is  $\tilde{\Delta}\Psi \equiv -d^\dagger d\Psi$ , and where we have used  $\tilde{*}dd^\dagger = d^\dagger d\tilde{*}$ . Equation (15) is the coordinate-invariant version of the

RW equation. In fact, it generalizes the RW equation, since it is not restricted to perturbations of static background configurations. (The fact that the RW function  $\Psi \equiv R^3 \tilde{*}d(R^{-2}H)$  and the scalar potential  $\Phi$ , defined by  $H = \tilde{*}d(R\Phi)$ , satisfy the same equation will be explained at the end of Sect. VB.)

The positivity of the RW potential for  $\ell \neq 1$  follows from the general expression (2) for  $G_{ab}$ , which yields the coordinate-independent vacuum background equation  $R\tilde{\Delta}R + \langle dR, dR \rangle = 1$ . By virtue of this, Eq. (15) assumes the form

$$\left[ -N\tilde{\Delta} + V_{RW} \right] \Psi = 0,$$

with

$$V_{RW} \equiv \frac{N}{R^2} [3(N-1) + \ell(\ell+1)]$$

and  $N \equiv \langle dR, dR \rangle$ . Hence,  $V_{RW}$  is positive for finite values of  $R$  if  $dR$  is space-like and  $\ell \geq 2$ .

We may now use standard Schwarzschild coordinates  $r$  and  $t$ , defined by

$$R(r, t) = r, \quad \tilde{g} = -N S^2 dt^2 + \frac{1}{N} dr^2, \quad (16)$$

to cast the RW equation into its well-known form. For a Schwarzschild background with mass  $M$  we have  $N(r) = 1 - 2M/r$ ,  $S(r) = 1$ ,  $N\tilde{\Delta} = -\partial_t^2 + N\partial_r N\partial_r$ , and thus

$$\left[ \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial r_*^2} + \frac{N}{r^2} \left( \ell(\ell+1) - \frac{6M}{r} \right) \right] \Psi = 0, \quad (17)$$

with  $dr_* \equiv N^{-1}dr$ . For  $\ell \geq 2$  the potential is non-negative in the domain of outer communications, and vanishes only asymptotically. Therefore, Eq. (17) admits no unstable dynamical modes. Furthermore, well-behaved stationary modes with  $\ell \geq 2$  can also be excluded in a rigorous manner by applying the argument given in Appendix B.

It remains to discuss the perturbations with  $\ell = 1$ , for which Eq. (15) is immediately seen to admit the solution  $1/R$ . Since  $\lambda = 0$ , we may also directly integrate Eq. (14), which yields

$$d \left( \frac{H}{R^2} \right) = a \frac{6M}{R^4} \tilde{*}1,$$

where  $6aM$  is a constant of integration. At this point it is important to recall that for  $\ell = 1$  the one-form  $H \equiv h$  is *not* coordinate-invariant, but transforms according to  $H \rightarrow H + R^2 d(f/R^2)$ . This implies that the solution of the homogeneous part of the above equation is a pure gauge. Hence, with respect to Schwarzschild coordinates, the only admissible solution of the perturbation equations (stationary and non-stationary) is  $H = 2a(M/r)dt$ . Using  $S_\theta^{\ell=1} = 0$  and  $S_\varphi^{\ell=1} = -\sin^2\theta$ , one finds with Eq. (3)

$$\delta g_{t\varphi} = -a \frac{2M}{r} \sin^2 \vartheta,$$

which describes the Kerr metric in first order of the rotation parameter  $a$ . In conclusion, we have established the well-known result that the only physically admissible odd-parity *vacuum* perturbation of the Schwarzschild metric lies in the sector  $\ell = 1$  and describes the stationary Kerr mode.

### III. PERTURBATIONS OF YANG-MILLS FIELDS

We are interested in perturbations of spherically symmetric EYM solitons and black holes which give rise to odd-parity metric excitations. Before deriving the gauge- and coordinate-invariant expressions for the stress-energy tensor and the YM equations, we briefly recall some features of the background configurations.

#### A. Einstein-Yang-Mills Background configurations

The spherically symmetric EYM background configurations are assumed to be purely magnetic [19], but not necessarily static. The metric is given by Eq. (1), while the gauge potential is parametrized in terms of a scalar field  $w(x^b)$  on  $\tilde{M}$ ,

$$A = (1 - w) \hat{*} d\tau_r, \quad (18)$$

where  $\tau_r \equiv \underline{\tau} \cdot \underline{e}_r$ . Here the  $\tau_k \equiv \sigma_k/(2i)$  are the  $\mathfrak{su}(2)$  generators,  $\underline{e}_r$  is the radial unit vector in  $\mathbb{R}^3$ , and the  $\sigma_k$  are the constant Cartesian Pauli-matrices. The total exterior derivative of the vector valued function  $\underline{e}_r$  is  $\hat{\theta}^A \underline{e}_A$  (with  $A = \vartheta, \varphi$ ), implying that

$$d\tau_r = \tau_\vartheta d\vartheta + \tau_\varphi \sin\vartheta d\varphi.$$

(See Appendix D for details.) Since  $\tau_r$  is an eigenfunction of the spherical Laplacian  $d\hat{*}d\tau_r = -2\tau_r d\Omega$ , the background field strength,  $F = dA + A \wedge A$ , becomes

$$F = -dw \wedge \hat{*}d\tau_r + (w^2 - 1)\tau_r d\Omega. \quad (19)$$

Using this expression, the components of the stress-energy tensor,  $T_{\mu\nu} = \frac{1}{4\pi} \text{Tr} \{ F_{\mu\alpha} F_\nu^\alpha - \frac{1}{4} g_{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} \}$ , with respect to the background metric (1) become

$$T_{ab} = \frac{1}{4\pi R^2} \left[ 2w_a w_b - \frac{1}{2} \tilde{g}_{ab} \left( 2w_c w^c + \frac{(w^2 - 1)^2}{R^2} \right) \right],$$

$$T_{AB} = \frac{1}{4\pi R^2} g_{AB} \frac{(w^2 - 1)^2}{2R^2}, \quad T_{Ab} = 0, \quad (20)$$

where  $w_a \equiv \tilde{\nabla}_a w$ , and where  $\text{Tr} \{ \}$  denotes the normalized trace,  $\text{Tr} \{ \tau_i^2 \} = 1$ .

The background YM equation,  $D\hat{*}F \equiv d\hat{*}F + [A, \hat{*}F] = 0$ , is obtained from the expression  $\hat{*}F = -\tilde{*}dw \wedge d\tau_r +$

$R^{-2}(w^2 - 1)\tau_r \tilde{*}1$ , using the fact that  $d\tau_r$  commutes with  $\hat{*}d\tau_r$ , and  $[d\tau_r, \tau_r] = \hat{*}d\tau_r$ . One finds

$$\tilde{\Delta}w = w \frac{w^2 - 1}{R^2}, \quad (21)$$

where  $d^\dagger = \tilde{*}d\tilde{*}$ , and  $\tilde{\Delta}w = -d^\dagger dw = \tilde{\nabla}^a \tilde{\nabla}_a w$ . The Einstein equations,  $G_{\mu\nu} = 8\pi G T_{\mu\nu}$ , are obtained from the formulas (2) and (20). Also using  $T_\mu^\mu = 0$ , one finds

$$G \frac{2}{R^2} \left[ 2w_a w_b - \frac{1}{2} \tilde{g}_{ab} \left( 2w_c w^c + \frac{(w^2 - 1)^2}{R^2} \right) \right], \quad (22)$$

$$1 - \frac{1}{2} \tilde{\Delta}(R^2) = G \frac{(w^2 - 1)^2}{R^2}. \quad (23)$$

Equations (21) - (23) are the spherically symmetric EYM equations in coordinate-invariant form. In the static case we may evaluate these expressions for the metric (16), which yields (a prime denoting the derivative with respect to  $r$ )

$$\frac{1}{S} (NSw')' = w \frac{w^2 - 1}{r^2} \quad (24)$$

for the YM equation (21), and, with  $N(r) \equiv 1 - 2m(r)/r$ ,

$$m' = \frac{G}{2} \left[ \frac{(w^2 - 1)^2}{r^2} + 2N(w')^2 \right], \quad (25)$$

$$\frac{S'}{S} = 2G \frac{(w')^2}{r} \quad (26)$$

for Eq. (23) and for the trace-free part of Eq. (22), respectively. Two special Abelian solutions to Eqs. (24) - (26) are the Schwarzschild metric,  $m(r) = M = \text{constant}$ ,  $S = 1$ ,  $w = 1$ , and the RN metric with mass  $M$  and unit magnetic charge,  $N = 1 - 2M/r + G/r^2$ ,  $S = 1$ ,  $w = 0$ .

Asymptotically flat *non-Abelian* solutions with finite energy and nontrivial gauge fields are the solitons found by Bartnik and McKinnon [1], and the corresponding black holes with hair [2]. They are obtained by numerical methods and by analyzing the local solutions at the singular points of Eqs. (24)-(26), that is, at the origin,  $r = 0$ , the horizon,  $N(r_H) = 0$ , and at infinity,  $r = \infty$ . The local background solutions are given in Appendix C, since their behavior will be crucial to the existence of regular singular points of the perturbation equations.

#### B. Gauge- and coordinate-invariant Yang-Mills perturbations

In Appendix D we construct a convenient basis of  $\mathfrak{su}(2)$ -valued spherical harmonic one-forms. The odd-parity perturbations of the YM potential are then given in terms of two one-forms,  $\alpha$  and  $\beta$ , and three scalar fields,  $\mu$ ,  $\nu$  and  $\sigma$ , over  $\tilde{M}$ ,

$$\delta A^{(\ell > 1)} = X_1 \alpha + X_2 \beta + \mu \tau_r dY + \nu Y d\tau_r + \sigma \hat{\nabla} X_2, \quad (27)$$

where  $X_1$ ,  $X_2$ , and  $X_3$  are a scalar basis of  $\mathfrak{su}(2)$ -valued spherical harmonics,

$$X_1 = Y \tau_r, \quad X_2 = \hat{g}^{AB} \tau_A \hat{\nabla}_B Y, \quad X_3 = \hat{\eta}^{AB} \tau_A \hat{\nabla}_B Y,$$

while  $Y \equiv Y^{\ell m}$  denote the ordinary spherical harmonics. (The antisymmetric tensor  $\hat{\eta}_{AB}$  is defined by  $\hat{\eta}^A_B = \hat{\eta}^A_B \hat{\theta}^B$ .) As usual, the cases  $\ell = 1$  and  $\ell = 0$  must be treated separately: For  $\ell = 1$ , one has  $\hat{\nabla} X_2^{(\ell=1)} = -Y^{(\ell=1)} d\tau_r$ , implying that  $\nu$  and  $\sigma$  combine to a single amplitude. Hence,

$$\delta A^{(\ell=1)} = X_1 \alpha + X_2 \beta + \mu \tau_r dY + \nu Y d\tau_r. \quad (28)$$

In contrast to the gravitational sector, the odd-parity YM sector is not empty for  $\ell = 0$ . As  $Y^{(\ell=0)}$  is constant,  $\delta A$  is parametrized in terms of the one-form  $\alpha$  and the function  $\nu$ ,

$$\delta A^{(\ell=0)} = \tau_r \alpha + \nu d\tau_r. \quad (29)$$

One may now study the behavior of  $\delta A$  under gauge transformations,  $\delta A \rightarrow \delta A + D\chi$ , and under coordinate transformations,  $\delta A \rightarrow \delta A + \mathcal{L}_X A$ . Here  $D$  is the gauge covariant derivative with respect to the background connection (18),  $\chi$  is an  $\mathfrak{su}(2)$ -valued scalar field with odd parity, and  $\mathcal{L}_X$  is the Lie derivative with respect to the infinitesimal vector field  $X$  defined in Eq. (4). Considering both gauge and coordinate transformations, the following results are established in Appendix E:

For  $\ell > 1$  the metric perturbations are originally parametrized in terms of the function  $\kappa$  and the one-form  $h$ , while the YM amplitudes are given by two one-forms,  $\alpha$  and  $\beta$ , and three functions,  $\mu$ ,  $\nu$  and  $\sigma$ . Using the complete gauge and coordinate freedom, the entire set of perturbations reduces to three one-forms,  $H$ ,  $A$  and  $B$ , and one function,  $C$ , all of which are invariant under both coordinate and gauge transformations. Adopting the ODG ( $\kappa = 0$ ) and the YM gauge  $\mu = \sigma = 0$ , the quantities  $H$ ,  $A$ ,  $B$ , and  $C$ , coincide with the original amplitudes  $h$ ,  $\alpha$ ,  $\beta$ , and  $\nu$ . [See Eqs. (E6) and (E12).] Hence, all physically relevant perturbations with  $\ell > 1$  are given by

$$\begin{aligned} \delta g_{AB}^{(\ell>1)} &= \delta g_{ab}^{(\ell>1)} = 0, \quad \delta g_{Ab}^{(\ell>1)} = H_b S_A, \\ \delta A^{(\ell>1)} &= X_1 A + X_2 B + C Y d\tau_r, \end{aligned} \quad (30)$$

with gauge- and coordinate-invariant amplitudes  $H$ ,  $A$ ,  $B$ , and  $C$ . The ODG for the metric perturbations, together with the YM gauge  $\mu = \sigma = 0$  will be called the off-diagonal standard gauge (ODSG) henceforth. *In the ODSG all gravitational and YM perturbations coincide with the corresponding coordinate- and gauge-invariant quantities.*

For  $\ell = 1$  the metric perturbations are already off-diagonal and there exists a gauge for which the YM scalars  $\mu$  and  $\nu$  vanish, and the remaining amplitudes,  $\alpha$  and  $\beta$ , coincide with the two gauge-invariant one-forms

$a$  and  $b$ , defined in Eq. (E4). The perturbations are therefore given by

$$\begin{aligned} \delta g_{AB}^{(\ell=1)} &= \delta g_{ab}^{(\ell=1)} = 0, \quad \delta g_{Ab}^{(\ell=1)} = h_b S_A, \\ \delta A^{(\ell=1)} &= X_1 a + X_2 b, \end{aligned} \quad (31)$$

where  $a$  and  $b$  are gauge-invariant, but neither the metric nor the YM perturbations are invariant under coordinate transformations. The linearized EYM equations involve, however, only the gauge- and coordinate-invariant combinations

$$\bar{a} \equiv a + \frac{h}{R^2}, \quad \bar{b} \equiv b + w \frac{h}{R^2}, \quad (32)$$

and  $d(R^{-2}h)$ , as we shall see later.

For  $\ell = 0$  there exist no metric perturbations in the odd-parity sector, and the YM perturbations are comprised within a single gauge-invariant one-form  $a$ , defined in Eq. (E8),

$$\delta g_{\mu\nu}^{(\ell=0)} = 0, \quad \delta A^{(\ell=0)} = \tau_r a. \quad (33)$$

#### IV. THE PERTURBATION EQUATIONS

In this section we give the equations governing the odd-parity perturbations of a spherically symmetric soliton or black hole EYM background configuration. The amplitudes are parametrized in terms of the gravitational one-form  $H$ , the YM one-forms  $A$ ,  $B$ , and the YM scalar  $C$ . All amplitudes are gauge- and coordinate-invariant and, as we are not introducing specific coordinates, the resulting equations are not limited to static background configurations. The derivations are considerably simplified by adopting the ODSG and by taking advantage of the  $\mathfrak{su}(2)$  harmonics constructed in Appendix D. However, as the computations are still lengthy, we discuss only the basic steps in sections IV A, IV B and IV C for  $\ell > 1$ ,  $\ell = 1$  and  $\ell = 0$ , respectively, and give a self-contained compilation of the results in Sect. IV D.

##### A. Equations for $\ell > 1$

For  $\ell > 1$  we may proceed in the ODSG for which the metric and the YM perturbations coincide with the gauge- and coordinate-invariant amplitudes  $H$ ,  $A$ ,  $B$ , and  $C$ :

$$\begin{aligned} \delta g_{Ab}^{(\ell>1)} &= H_b S_A, \\ \delta A^{(\ell>1)} &= X_1 A + X_2 B + C Y d\tau_r. \end{aligned} \quad (34)$$

We start by computing the coordinate-invariant stress-energy tensor. According to Eqs. (8) and (9) we have

$$\delta T_{ab}^{inv} = \delta T_{ab}^{ODG}, \quad \delta T_{AB}^{inv} = \delta T_{AB}^{ODG}, \quad (35)$$

and

$$\delta T_{Ab}^{inv} = \delta T_{Ab}^{ODG} - H_b T_A^B S_B, \quad (36)$$

since  $\kappa = 0$  and  $H_a = h_a$  in the ODG. The  $\delta T_{\mu\nu}$  consist of perturbations arising from variations with respect to the metric and the YM fields,  $\delta T_{\mu\nu} = \delta_g T_{\mu\nu} + \delta_A T_{\mu\nu}$ , where

$$\begin{aligned} \delta_g T_{\mu\nu} = & -\frac{1}{4\pi} \text{Tr} \left\{ \frac{1}{4} F_{\alpha\beta} F^{\alpha\beta} \delta g_{\mu\nu} \right. \\ & \left. + \left( F_\mu^\alpha F_\nu^\beta - \frac{1}{2} g_{\mu\nu} F_\gamma^\alpha F^{\beta\gamma} \right) \delta g_{\alpha\beta} \right\} \end{aligned}$$

and

$$\delta_A T_{\mu\nu} = \frac{1}{4\pi} \text{Tr} \left\{ F_\nu^\alpha \delta F_{\alpha\mu} + F_\mu^\alpha \delta F_{\alpha\nu} - \frac{1}{2} g_{\mu\nu} F^{\alpha\beta} \delta F_{\alpha\beta} \right\}.$$

In the ODSG the linearized field strength,  $\delta F = D\delta A$ , is obtained from the formula (34) for  $\delta A^{(\ell>1)}$ . Recalling that  $D$  is the gauge covariant derivative with respect to the background potential (18), one finds, also using the identities (D3),

$$\begin{aligned} \delta F^{(\ell>1)} = & X_1 dA + X_2 dB - X_3 C d\Omega - B \wedge \hat{\nabla} X_2 \\ & + (wB - A) \wedge \tau_r dY + (dC - wA) \wedge Y d\tau_r. \end{aligned} \quad (37)$$

Using this, as well as the expression (19) for the background field strength  $F$  and the formulas (34) for the metric perturbations, we end up with

$$\delta T_{ab}^{inv} = 0, \quad \delta T_{AB}^{inv} = \frac{1}{4\pi} \langle B, dw \rangle 2\hat{\nabla}_{\{A} S_{B\}}, \quad (38)$$

and

$$\begin{aligned} & \delta T_{Ab}^{inv} \\ = & \frac{S_A}{4\pi R^2} \left[ (w^2 - 1)(A_b - wB_b) + R^2 (dB)_{ba} w^a - C w_b \right. \\ & \left. + \langle H, dw \rangle w_b - \left( \langle dw, dw \rangle + \frac{(w^2 - 1)^2}{R^2} \right) H_b \right], \end{aligned} \quad (39)$$

where we recall that all amplitudes are gauge- and coordinate-invariant. Here and in the following we use the obvious notations  $w^a \equiv \tilde{g}^{ab} \tilde{\nabla}_b w$  and  $\langle, \rangle$  for the inner product with respect to  $\tilde{g}$ , e.g.,  $\langle H, dw \rangle \equiv \tilde{g}^{ab} H_a \tilde{\nabla}_b w$ . [There is no factor 1/2 in front of the last term in Eq. (39), since, according to Eqs. (20) and (36),  $\delta T_{Ab}^{inv}$  and  $\delta T_{Ab}$  differ by the term  $(8\pi)^{-1} R^{-4} (w^2 - 1)^2 H_b S_A$  in the ODG.]

The Einstein equations,  $\delta G_{\mu\nu}^{inv} = 8\pi G \delta T_{\mu\nu}^{inv}$ , are now obtained from the above expressions and the formulae (12) and (13) for  $\delta G_{\mu\nu}^{inv}$ . We have already argued that the  $(AB)$ -equation,

$$d^\dagger H = -4G \langle B, dw \rangle, \quad (40)$$

is a consequence of the  $(Ab)$ -equations and the linearized Bianchi identity. While this was obvious for vacuum perturbations, one now needs the YM equations given below

to verify this fact. Hence, the only independent Einstein equation is the one for the coordinate-invariant metric one-form  $H$ ,

$$\begin{aligned} & d^\dagger \left[ R^4 d \left( \frac{H}{R^2} \right) \right] + \lambda H \\ = & 4G(w^2 - 1) \left[ A - wB - \frac{w^2 - 1}{R^2} H \right] \\ & + 4G [\tilde{*}(R^2 dB + dw \wedge H) \tilde{*}dw - Cdw], \end{aligned} \quad (41)$$

where we also recall that  $\lambda \equiv (\ell - 1)(\ell + 2)$ . Here we have used the identities  $\langle H, dw \rangle dw - \langle dw, dw \rangle H = \tilde{*}(dw \wedge H) \tilde{*}dw$  and  $(dB)_{ab} w^b dx^a = (\tilde{*}dB) \tilde{*}dw$ .

The linearized YM equations also involve perturbations of both YM and metric fields. The latter arise from the variation of the Hodge dual in  $\delta(D * F) = 0$ , and yield the terms on the RHS of the following general expression:

$$D * \delta F + [\delta A, *F] = 2 D * \mathcal{F} - d \left( \frac{\delta \sqrt{-\tilde{g}}}{\sqrt{-\tilde{g}}} \right) \wedge *F,$$

where  $\mathcal{F}_{\mu\nu} \equiv F_{[\mu}^\sigma \delta g_{\nu]\sigma}$ . Since the dual of this is an equation between one-forms, and since the odd-parity basis of one-forms is five-dimensional for  $\ell > 1$ , we obtain five equations. Again, the computation is considerably simplified in the ODSG for which we may use the gauge-invariant perturbations given in Eqs. (34). As expected, it turns out that two YM equations can be obtained from the remaining ones. Using the tools developed in Appendix D, we eventually end up with the following set of equations for the one-forms  $A$ ,  $B$  and the scalar  $C$ :

$$\begin{aligned} & d^\dagger (R^2 dA) + [\lambda + 2(w^2 + 1)] A - 2[\lambda + 2] wB \\ & - 2wdC + 2Cdw = (\lambda + 2) \frac{w^2 - 1}{R^2} H, \end{aligned} \quad (42)$$

$$\begin{aligned} & d^\dagger (R^2 dB) - 2wA + [\lambda + (w^2 + 1)] B + dC \\ = & d^\dagger (H \wedge dw) - w \frac{w^2 - 1}{R^2} H, \end{aligned} \quad (43)$$

$$C = R^2 d^\dagger B - \langle dw, H \rangle. \quad (44)$$

The remaining two YM equations are the integrability conditions for Eqs. (42) and (43). Also using Eq. (44), these become

$$d^\dagger \left[ A + \frac{1 - w^2}{R^2} H \right] = -2 \langle B, dw \rangle, \quad (45)$$

and

$$\begin{aligned} & \tilde{\Delta} C - [\lambda + (w^2 + 1)] \frac{C}{R^2} \\ = & 2 \langle A, dw \rangle - wd^\dagger A + [\lambda + 2] \frac{\langle dw, H \rangle}{R^2}. \end{aligned} \quad (46)$$

Since Eqs. (40), (45) and (46) are consequences of the remaining equations, the complete system of perturbation equations consists of the three coupled equations (41), (42) and (43) for the three gauge- and coordinate-invariant one-forms  $A$ ,  $B$  and  $H$ , where  $C$  is given by Eq. (44). It will also turn out to be convenient to write these equations in terms of the one-forms  $\bar{A}$  and  $\bar{B}$ , defined by

$$\bar{A} \equiv A + \frac{H}{R^2}, \quad \bar{B} \equiv B + w \frac{H}{R^2}, \quad (47)$$

in terms of which Eqs. (41), (42) and (43) assume the form

$$\begin{aligned} & d^\dagger (R^4 F_H) + \lambda H \\ &= 4G [(R^2 \tilde{*} F_B) \tilde{*} dw - C dw + (w^2 - 1)(\bar{A} - w \bar{B})], \end{aligned} \quad (48)$$

$$\begin{aligned} & d^\dagger (R^2 F_A) + \lambda \left( \bar{A} - 2w \bar{B} + w^2 \frac{H}{R^2} \right) \\ &= -2(w^2 + 1)\bar{A} + 4w \bar{B} - 2C dw + 2w dC, \end{aligned} \quad (49)$$

$$\begin{aligned} & d^\dagger (R^2 F_B) + \lambda \left( \bar{B} - w \frac{H}{R^2} \right) \\ &= 2w \bar{A} - (w^2 + 1)\bar{B} - dC, \end{aligned} \quad (50)$$

with

$$C = R^2 \left[ d^\dagger \bar{B} - w d^\dagger \left( \frac{H}{R^2} \right) \right]. \quad (51)$$

Here we have introduced the two-forms  $F_A$ ,  $F_B$  and  $F_H$ , which are defined in terms of  $H$ ,  $\bar{A}$  and  $\bar{B}$  as follows:

$$\begin{aligned} F_A &\equiv d\bar{A} - F_H, \quad F_B \equiv d\bar{B} - w F_H, \\ F_H &\equiv d \left( \frac{H}{R^2} \right), \end{aligned} \quad (52)$$

i.e.,  $F_A = dA$ ,  $F_B = dB + dw \wedge R^{-2}H$ . The three equations (48)–(50) for the invariant one-forms  $\bar{A}$ ,  $\bar{B}$  and  $H$ , with  $C$  according to Eq. (51), govern all physical odd-parity perturbations with  $\ell > 1$ . We shall now argue that these equations hold for  $\ell = 1$  as well, provided that one sets  $C = 0$ .

## B. Equations for $\ell = 1$

For  $\ell = 1$  the metric perturbations are off-diagonal and described by the one-form  $h$ , while the YM potential is parametrized in terms of two one-forms  $a$  and  $b$ ,

$$\delta g_{Ab}^{(\ell=1)} = h_b S_A, \quad \delta A^{(\ell=1)} = X_1 a + X_2 b. \quad (53)$$

Although  $a$  and  $b$  are gauge-invariant, they are not invariant under coordinate transformations, and neither is  $h$ . As the linearized YM and Einstein equations are coordinate-invariant, these will only involve the gauge-

and coordinate-invariant one forms  $\bar{a}$  and  $\bar{b}$  defined in Eq. (32).

The perturbation equations for  $\ell = 1$  are obtained from the equations for  $\ell > 1$  as follows: The linearized field strength two-form,  $\delta F = D\delta A$ , for the background potential (18) and the perturbation (53) becomes

$$\begin{aligned} \delta F^{(\ell=1)} &= X_1 da + X_2 db \\ &+ (wb - a) \wedge \tau_r dY + (b - wa) \wedge Y d\tau_r. \end{aligned} \quad (54)$$

Formally, this is also obtained from the expression (37) for  $\delta F^{(\ell>1)}$  by substituting  $a$  for  $A$ ,  $b$  for  $B$  and by setting  $C = 0$ , where one also has to use  $\hat{\nabla} X_2^{(\ell=1)} = -Y^{(\ell=1)} d\tau_r$ ; see Appendix D for details. Hence, the invariant stress energy tensor for  $\ell = 1$  is obtained from the expressions (38) and (39) for  $\ell > 1$  by applying these substitutions and by replacing  $h$  for  $H$ . This yields

$$\delta T_{ab}^{inv} = 0, \quad \delta T_{AB}^{inv} = 0$$

and

$$\begin{aligned} \delta T_{Ab}^{inv} dx^b &= \frac{S_A}{4\pi R^2} [(w^2 - 1)(a - wb) + R^2(\tilde{*}db)\tilde{*}dw] \\ &+ \frac{S_A}{4\pi R^2} \left[ \tilde{*}(dw \wedge h)\tilde{*}dw - \frac{(w^2 - 1)^2}{R^2} h \right] S_A. \end{aligned}$$

The coordinate-invariance of the last expression becomes manifest by writing it in terms of the one forms  $\bar{a}$  and  $\bar{b}$  given in Eq. (32). One finds

$$\begin{aligned} \delta T_{Ab}^{inv} dx^b &= \frac{S_A}{4\pi R^2} (w^2 - 1)(\bar{a} - w\bar{b}) \\ &+ \frac{S_A}{4\pi} \tilde{*} \left[ d\bar{b} - w d \left( \frac{h}{R^2} \right) \right] \tilde{*} dw, \end{aligned} \quad (55)$$

where the metric perturbation enters only via the invariant two-form  $d(R^{-2}h)$ .

It is now obvious that the complete set of linearized EYM equations in terms of the gauge and coordinate-invariant amplitudes  $\bar{a}$  and  $\bar{b}$  is obtained from Eqs. (48)–(50), by substituting  $\bar{A} \rightarrow \bar{a}$ ,  $\bar{B} \rightarrow \bar{b}$  and  $C \rightarrow 0$ . As we also have to substitute  $H \rightarrow h$ , the LHS of Eqs. (48)–(50) would, at a first glance, involve the non-coordinate-invariant amplitude  $h$ . However, since  $\lambda = 0$  for  $\ell = 1$ , the terms involving  $h$  itself vanish identically. We also point out that the algebraic equation (51) for  $C$  is not present for  $\ell = 1$ , because the basis of one-forms is reduced by one dimension. The complete set of perturbation equations in the sector  $\ell = 1$  thus assumes the surprisingly simple form

$$\begin{aligned} & d^\dagger (R^4 F_h) - 4G [(R^2 \tilde{*} F_b) \tilde{*} dw + (w^2 - 1)(\bar{a} - w\bar{b})] = 0, \\ & d^\dagger (R^2 F_a) + 2(w^2 + 1)\bar{a} - 4w\bar{b} = 0, \\ & d^\dagger (R^2 F_b) - 2w\bar{a} + (w^2 + 1)\bar{b} = 0, \end{aligned} \quad (56)$$

where  $\bar{a}$  and  $\bar{b}$  are the gauge- and coordinate-invariant one-forms given in Eq. (32), in terms of which the two-forms  $F_h$ ,  $F_a$  and  $F_b$  are defined by



$$F_h \equiv d\left(\frac{h}{R^2}\right), \quad F_a \equiv d\bar{a} - F_h, \quad F_b \equiv d\bar{b} - wF_h.$$

### C. Equations for $\ell = 0$

For  $\ell = 0$  there exist no metric perturbations with odd parity, and  $\delta A$  can be expressed in terms of a single gauge-invariant one-form

$$\delta g_{\mu\nu}^{(\ell=0)} = 0, \quad \delta A^{(\ell=0)} = \tau_r a.$$

The field strength is obtained by setting  $Y = 1$ ,  $B = 0$ ,  $C = 0$  in the expression (37) for  $\delta F^{(\ell>1)}$ , and by substituting  $a$  for  $A$ ,

$$\delta F^{(\ell=0)} = \tau_r da - wa \wedge d\tau_r.$$

The correct perturbation equation is now obtained from Eq. (49) by setting  $\bar{B} = C = H = 0$ , where  $\lambda = -2$  for  $\ell = 0$ . Also substituting  $a$  for  $\bar{A}$ , Eqs. (49) and (52) yield

$$d^\dagger(R^2 F_a) + 2w^2 a = 0, \quad \text{with } F_a \equiv da. \quad (57)$$

### D. Summary

All odd-parity perturbations of spherically symmetric, not necessarily static EYM configurations are governed by the three equations

$$d^\dagger(R^4 F_H) + \lambda H = 4G[(R^2 \tilde{*} F_B) \tilde{*} dw - Cdw + (w^2 - 1)(\bar{A} - w\bar{B})], \quad (58)$$

$$d^\dagger(R^2 F_A) + \lambda \left( \bar{A} - 2w\bar{B} + w^2 \frac{H}{R^2} \right) = -2(w^2 + 1)\bar{A} + 4w\bar{B} - 2Cdw + 2wdC, \quad (59)$$

$$d^\dagger(R^2 F_B) + \lambda \left( \bar{B} - w \frac{H}{R^2} \right) = 2w\bar{A} - (w^2 + 1)\bar{B} - dC, \quad (60)$$

for the three gauge- and coordinate-invariant one-forms  $\bar{A}$ ,  $\bar{B}$ , and  $H$ , where  $\lambda \equiv (\ell - 1)(\ell + 2)$ ,

$$C \equiv (1 - \delta_0^\ell - \delta_1^\ell) R^2 \left[ d^\dagger \bar{B} - wd^\dagger \left( \frac{H}{R^2} \right) \right], \quad (61)$$

and

$$F_H \equiv d\left(\frac{H}{R^2}\right), \quad F_A \equiv d\bar{A} - F_H, \quad F_B \equiv d\bar{B} - wF_H.$$

The above equations are valid for all values of  $\ell$ , where only Eq. (59) with  $H = \bar{B} = 0$  is present for  $\ell = 0$ .

However, the expressions for the gauge- and coordinate-invariant amplitudes in terms of the original metric and YM perturbations are different for  $\ell > 1$ ,  $\ell = 1$  and  $\ell = 0$ , respectively; see Appendix E.

$\ell > 1$ : The original metric perturbations are described by a one-form  $h$  and a function  $\kappa$ , while the YM perturbations are given in terms of two one-forms,  $\alpha$  and  $\beta$ , and three functions,  $\mu$ ,  $\nu$  and  $\sigma$ :

$$\delta g_{ab} = 0, \quad \delta g_{Ab} dx^b = h S_A, \quad \delta g_{AB} = 2\kappa \hat{\nabla}_{\{A} S_{B\}}, \\ \delta A = X_1 \alpha + X_2 \beta + \mu \tau_r dY + \nu Y d\tau_r + \sigma \hat{\nabla} X_2.$$

In terms of these amplitudes the invariant quantities appearing in Eqs. (58)-(60) are, according to Appendix E,

$$\bar{A} \equiv \alpha + \frac{h}{R^2} - d\left(\mu + w\sigma + w^2 \frac{\kappa}{R^2}\right), \\ \bar{B} \equiv \beta + w \frac{h}{R^2} - d\left(\sigma + w \frac{\kappa}{R^2}\right), \\ H \equiv h - R^2 d\left(\frac{\kappa}{R^2}\right). \quad (62)$$

$\ell = 1$ : The original metric perturbations are described by the one-form  $h$ , while the YM perturbations are given in terms of two one-forms,  $\alpha$  and  $\beta$ , and two functions,  $\mu$  and  $\nu$ :

$$\delta g_{ab} = 0, \quad \delta g_{Ab} dx^b = h S_A, \quad \delta g_{AB} = 0, \\ \delta A = X_1 \alpha + X_2 \beta + \mu \tau_r dY + \nu Y d\tau_r. \quad (63)$$

The invariant quantities now are

$$\bar{A} \equiv \alpha + \frac{h}{R^2} - d\left(\frac{\mu - w\nu}{1 - w^2}\right), \\ \bar{B} \equiv \beta + w \frac{h}{R^2} - d\left(\frac{w\mu - \nu}{1 - w^2}\right), \\ H \equiv h. \quad (64)$$

$\ell = 0$ : There exist no metric perturbations, and the YM perturbations are given in terms of a one-form,  $\alpha$ , and a function,  $\nu$ :

$$\delta A = \tau_r \alpha + \nu d\tau_r.$$

In terms of  $\alpha$  and  $\nu$  the invariant quantity  $\bar{A}$  is given by

$$\bar{A} \equiv \alpha - d\left(\frac{\nu}{w}\right), \quad (65)$$

and, as mentioned above, the correct perturbation equation is Eq. (59) with  $\lambda = -2$ ,  $C = 0$ ,  $\bar{B} = 0$  and  $H = 0$ .

## V. NON-ABELIAN STABILITY AND LOCAL UNIQUENESS OF THE REISSNER-NORDSTRÖM SOLUTION

For  $w \equiv 0$  the static, spherically symmetric EYM equations (24)-(26) admit the RN solution with unit magnetic

charge. The stability and local uniqueness properties of the RN metric with respect to non-Abelian perturbations are, therefore, obtained from Eqs. (58)-(60), which decouple into two sets for  $w \equiv 0$ : The first set, involving the one-forms  $H$  and  $\bar{A}$  only, is obtained from Eqs. (58) and (59),

$$\begin{aligned} d^\dagger \left[ R^4 d \left( \frac{H}{R^2} \right) \right] + \lambda H &= -4G\bar{A}, \\ d^\dagger \left[ R^2 d \left( \bar{A} - \frac{H}{R^2} \right) \right] + (\lambda + 2)\bar{A} &= 0 \text{ for } \ell \geq 1. \end{aligned} \quad (66)$$

Since  $w \equiv 0$ , the remaining equation for  $\bar{B}$  does not contain the amplitudes  $H$  and  $\bar{A}$ . Using  $C = R^2 d^\dagger \bar{B}$  for  $\ell > 1$  and  $C = 0$  for  $\ell = 1$ , we have

$$d^\dagger (R^2 d \bar{B}) + d (R^2 d^\dagger \bar{B}) + (\lambda + 1)\bar{B} = 0 \text{ for } \ell > 1, \quad (67)$$

$$d^\dagger (R^2 d \bar{B}) + \bar{B} = 0 \text{ for } \ell = 1. \quad (68)$$

Since  $\bar{A}$  is the gauge- and coordinate-invariant version of the amplitude in front of the isospin harmonics  $\tau_r Y^\ell$ , Eqs. (66) govern the Abelian part of the perturbations, that is, Einstein-Maxwell perturbations of the RN metric. In contrast to this, Eqs. (67) and (68) for  $\bar{B}$  are not present in the Abelian case, and describe non-Abelian perturbations of the RN metric with  $\ell > 1$  and  $\ell = 1$ , respectively.

### A. Perturbations with $\ell > 1$

We start with Eqs. (66) describing the Abelian part of the perturbations. For  $\ell > 1$  the integrability conditions for these equations are  $d^\dagger \bar{A} = 0$  and  $d^\dagger H = 0$ , implying the existence of two scalar fields,  $\Psi_H$  and  $\Psi_A$ , defined by

$$\tilde{*}d(R\Psi_H) \equiv \sqrt{\lambda}H, \quad \tilde{*}d\Psi_A \equiv \sqrt{4G\lambda}\bar{A}.$$

Substituting  $\Psi_H$  and  $\Psi_A$  for  $H$  and  $\bar{A}$  in Eqs. (66), and integrating both equations yields the following coupled wave equations for the scalar fields  $\Psi_H$  and  $\Psi_A$ :

$$\begin{aligned} \tilde{\Delta}\Psi_H &= \left[ R d^\dagger \left( \frac{dR}{R^2} \right) + \frac{\lambda}{R^2} \right] \Psi_H + \frac{\sqrt{4G\lambda}}{R^3} \Psi_A, \\ \tilde{\Delta}\Psi_A &= \frac{\sqrt{4G\lambda}}{R^3} \Psi_H + \left[ \frac{\lambda + 2}{R^2} + \frac{4G}{R^4} \right] \Psi_A. \end{aligned}$$

For  $w = 0$  the background equation (23) becomes  $R^3 d^\dagger (dR/R^2) = 3\langle dR, dR \rangle - 1 + G/R^2$ . Using this and introducing standard Schwarzschild coordinates,  $R = r$ ,  $\langle dR, dR \rangle = N = 1 - 2M/r + G/r^2$ , yields

$$\begin{aligned} \left[ -\tilde{\Delta} + \frac{1}{r^2} \left( \lambda + 2 - \frac{3M}{r} + \frac{4G}{r^2} \right) \right] \begin{pmatrix} \Psi_H \\ \Psi_A \end{pmatrix} \\ + \frac{1}{r^3} \begin{pmatrix} -3M & \sqrt{4G\lambda} \\ \sqrt{4G\lambda} & 3M \end{pmatrix} \begin{pmatrix} \Psi_H \\ \Psi_A \end{pmatrix} &= 0. \end{aligned} \quad (69)$$

The above equation was first obtained by Moncrief by different means [18]. Since the off-diagonal part of the potential is symmetric and constant, Eq. (69) can be decoupled. Using the non-negativity of  $N(r)$ , as well as the regularity condition  $M \geq G$ , the eigenvalues of the potential are found to be positive, implying the absence of unstable modes. Taking advantage of the argument presented in Appendix B, *stationary* modes are excluded as well. (The eigenvalues of the potential are positive for finite  $r$  and behave like  $\ell(\ell-1)r^{-2} + \mathcal{O}(r^{-3})$  for  $r \rightarrow \infty$ , implying that the asymptotically finite solutions behave like  $r^{-\ell}$ .) Hence, there exist neither unstable modes nor admissible stationary solutions to Eqs. (66) for  $\ell > 1$ .

In order to discuss the non-Abelian part of the perturbations we introduce the scalar fields  $\Pi_1 \equiv R^2 \tilde{*}d\bar{B}$  and  $\Pi_2 \equiv R^2 d^\dagger \bar{B}$ . In terms of these, Eq. (67) assumes the form

$$\tilde{*}d\Pi_1 + d\Pi_2 + (\lambda + 1)\bar{B} = 0, \quad (70)$$

which can also be viewed as the Hodge decomposition of the one-form  $\bar{B}$  (see the comments below). Applying the operators  $\tilde{*}d$  and  $d^\dagger \equiv \tilde{*}d\tilde{*}$  to this, it is immediately seen that  $\Pi_1$  and  $\Pi_2$  are subject to the same equation, namely

$$-\tilde{\Delta}\Pi_i + \frac{\lambda + 1}{R^2}\Pi_i, \quad i = 1, 2, \quad (71)$$

where we recall that  $\lambda = (\ell-1)(\ell+2)$ . With respect to the static RN background,  $R(r, t) = r$ ,  $\tilde{\mathbf{g}} = N(-dt^2 + dr_\star^2)$ , with  $N(r) = 1 - 2M/r + G/r^2$  and  $dr_\star = dr/N$ , one has

$$\left[ \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial r_\star^2} + N(r) \frac{\ell(\ell+1) - 1}{r^2} \right] \Pi_i = 0. \quad (72)$$

Since the operator is positive, unstable modes are absent. Furthermore, well-behaved stationary solutions are excluded as well, since the potential is of the type required to apply the argument given in Appendix B. [Also note that the one-form  $\bar{B}$  is obtained directly from  $\Pi_1$  and  $\Pi_2$  by Eq. (70).]

As we shall continue to use the above method, it is worthwhile noticing the following: In two dimensions an arbitrary one-form  $\theta$ , say, gives rise to two scalar fields,  $g_1 \equiv d^\dagger \theta$  and  $g_2 \equiv \tilde{*}d\theta$ . On the other hand, the Hodge decomposition of a one-form in two dimensions involves two different scalar fields,  $\theta \equiv df_1 + \tilde{*}df_2$  (provided that the harmonic part vanishes). If  $\theta$  is subject to a linear wave equation, then the latter gives rise to an *algebraic* relation between the two different parameterizations, although the two scalar pairs are defined on different differential levels. (This is also the reason why, in Sect. IID, we have obtained the same RW equation (15) for  $\Psi$  and  $\Phi$ , defined by  $\Psi = R^3 \tilde{*}d(H/R^2)$  and  $H = \tilde{*}d(R\Phi)$ , respectively.)

## B. Perturbations with $\ell = 1$

Defining  $\Pi_1 \equiv \tilde{*}R^2 d\bar{B}$  as for  $\ell > 1$ , Eq. (68) for  $\bar{B}$  reduces to  $\tilde{*}d\Pi_1 + \bar{B} = 0$ . Applying the operator  $\tilde{*}d$  yields the same equation for  $\Pi_1$  as before, that is, Eq. (71), where now  $\ell = 1$ . As the potential remains positive for  $\ell = 1$ , we conclude that Eq. (68) admits neither unstable modes nor admissible stationary perturbations, which establishes the stability and the local uniqueness of the RN metric with respect to non-Abelian odd-parity perturbations.

It remains to consider Eqs. (66) for  $\ell = 1$ , i.e., for  $\lambda = 0$ . As these equations are also present in the Abelian case, we will recover the absence of unstable modes, while the only stationary perturbations are those describing the Kerr-Newman excitations of the RN solution. This is seen as follows: For  $\lambda = 0$  the only integrability condition for Eqs. (66) is  $d^\dagger \bar{A} = 0$ . Using this to define the scalar field  $\Psi$  according to

$$\tilde{*}d\Psi \equiv \bar{A}, \quad (73)$$

Eqs. (66) can be integrated, which yields

$$R^4 \tilde{*}d \left( \frac{H}{R^2} \right) + 4G\Psi = 6Ma, \quad (74)$$

$$R^2 \left[ \tilde{\Delta}\Psi + \tilde{*}d \left( \frac{H}{R^2} \right) \right] - 2\Psi = 0, \quad (75)$$

where  $6Ma$  is a constant of integration, and where we have used the fact that  $\Psi$  is defined up to a constant in order to neglect the second constant of integration. Eliminating the gravitational perturbation  $H$  from the above equations yields the following inhomogeneous wave equation for  $\Psi$ :

$$\left[ -\tilde{\Delta} + \frac{2}{R^2} + \frac{4G}{R^4} \right] \Psi = \frac{6Ma}{R^4}. \quad (76)$$

As the operator on the LHS is positive, we conclude again that there are no unstable modes. Using standard Schwarzschild coordinates,  $R = r$ ,  $N = 1 - 2M/r + G/r^2$ , we have  $\tilde{\Delta} = -N^{-1}\partial_t^2 + \partial_r N \partial_r$ , and the inhomogeneous problem admits the particular solution  $\Psi = a/r$ . By virtue of Eq. (74) and definition (73) this yields, up to a gauge,

$$H = a(N - 1)dt, \quad \bar{A} = a \frac{N}{r^2} dt. \quad (77)$$

Recalling that for  $\ell = 1$  one has  $\delta g_{a\vartheta=0}$ ,  $\delta g_{a\varphi} = -H_a \sin^2\vartheta$  and  $\delta A = (\bar{A} - H/r^2)X_1 + (\bar{B} - wH/r^2)X_2$ , we find with  $w = 0$  and  $\bar{B} = 0$

$$\begin{aligned} \delta g_{t\varphi} &= a \left( \frac{2M}{r} - \frac{G}{r^2} \right) \sin^2\vartheta, \\ \delta A &= \frac{a}{r^2} \tau_r \cos\vartheta dt, \end{aligned}$$

which is the Kerr-Newman excitation of the magnetically charged RN metric. In order to see this, we compute the *electric* field,  $\delta E = -\delta F(\partial_t, \cdot) = -D\delta A(\partial_t, \cdot) = a\tau_r d(\cos\vartheta/r^2)$ , where we have used  $D\tau_r = 0$  and  $w = 0$ . Hence

$$\delta E = -\tau_r a \left( \frac{\sin\vartheta}{r^2} d\vartheta + \frac{2\cos\vartheta}{r^3} dr \right).$$

Since the magnetic field of the background solution is  $B = -\tau_r (*d\Omega)(\partial_t, \cdot) = -\tau_r (1/r^2) dr$  [see Eq. (19) for  $w = 0$ ], we obtain indeed the magnetically charged Kerr-Newman solution in first order of the rotation parameter  $a$ .

## C. Perturbations with $\ell = 0$

Since the odd-parity gravitational sector is empty for  $\ell = 0$ , the perturbations of the RN solution are governed by Eq. (59) with  $H = \bar{B} = 0$ ,  $w = 0$  and  $\lambda = -2$ ,

$$d\tilde{*}(R^2 d\bar{A}) = 0.$$

With respect to Schwarzschild coordinates the solution is  $\bar{A} = (q/r)dt$ , where  $q$  is a constant of integration. The perturbation of the gauge potential now becomes  $\delta A = \tau_r(q/r)dt$ , which gives rise to a radial electric field,

$$\delta E = -\tau_r \frac{q}{r^2} dr.$$

Hence, we obtain the embedded magnetic RN solution with infinitesimal electric charge  $q$ . (Note that the metric remains unchanged in first order of  $q$ .)

In conclusion, we have shown that the RN solution is stable with respect to both Abelian *and* non-Abelian odd-parity perturbations for all values of  $\ell$ . Also, the only physically admissible stationary modes are the Abelian ones, describing electric Kerr-Newman ( $\ell = 1$ ) and RN ( $\ell = 0$ ) excitations of the magnetic RN metric.

## VI. NON-ABELIAN STABILITY AND LOCAL UNIQUENESS OF THE SCHWARZSCHILD SOLUTION

The Schwarzschild metric solves the spherically symmetric EYM background equations with  $w = 1$ . As the stress-energy tensor is quadratic in the field strength, the gravitational perturbations decouple in first order for all values of  $\ell$ , and are governed by the RW equation for vacuum perturbations. The remaining equations, describing Abelian and non-Abelian perturbations of the Schwarzschild metric, admit no unstable modes, and, for  $\ell > 1$ , no acceptable stationary excitations either. For  $\ell = 1$  the only stationary YM perturbation is the RN mode in the Abelian sector.

### A. Perturbations with $\ell > 1$

The gauge- and coordinate-invariant one-forms  $\bar{A}$ ,  $\bar{B}$  and  $H$  given in Eqs. (62) for  $\ell > 1$  are well-defined for  $w = 1$ . The perturbations are, therefore, governed by Eqs. (58)-(60), where Eq. (58) decouples for  $w = 1$  and reduces to the usual equation describing the vacuum perturbations of the Schwarzschild metric,

$$d^\dagger \left[ R^4 d \left( \frac{H}{R^2} \right) \right] + \lambda H = 0. \quad (78)$$

In Sect. IID we have already recalled that this equation admits neither unstable nor well-behaved stationary solutions for  $\ell > 1$ .

In order to discuss the non-vacuum perturbations of the Schwarzschild metric, it is more convenient to resort to the original one-forms  $A = \bar{A} - H/R^2$  and  $B = \bar{B} - wH/R^2$ , used in Sect. IV to derive the perturbation equations. In terms of  $A$  and  $B$ , Eqs. (59) and (60) become for  $w = 1$

$$\begin{aligned} d^\dagger (R^2 dA) - 2d (R^2 d^\dagger B) + (\lambda + 4)A - 2(\lambda + 2)B &= 0, \\ d^\dagger (R^2 dB) + d (R^2 d^\dagger B) - 2A + (\lambda + 2)B &= 0. \end{aligned} \quad (79)$$

The above system is equivalent to four coupled equations for four scalar fields. In order to decouple these equations completely, we note the following: The terms with  $B$  and  $d^\dagger B$  can be eliminated, which shows that the integrability condition is  $d^\dagger A = 0$ . Using this, and applying the co-differential operator on either of the above equations, yields a wave equation for the scalar field  $d^\dagger B$  alone,

$$\left( -\tilde{\Delta} + \frac{\lambda + 2}{R^2} \right) \Pi_B = 0, \quad \Pi_B \equiv R^2 d^\dagger B. \quad (80)$$

Since the integrability condition implies that the scalar  $\Pi_A \equiv R^2 d^\dagger A$  vanishes, it remains to find the equations for the field strengths  $dA$  and  $dB$ , or, equivalently, for the scalar fields  $\Psi_A$  and  $\Psi_B$ , defined by

$$\Psi_A \equiv \tilde{*} R^2 dA, \quad \Psi_B \equiv \sqrt{\lambda + 2} \tilde{*} R^2 dB.$$

Applying the operator  $\tilde{*}d$  on Eqs. (79) then yields the system

$$\left[ -\tilde{\Delta} + \frac{1}{R^2} \begin{pmatrix} \lambda + 4 & -2\sqrt{\lambda + 2} \\ -2\sqrt{\lambda + 2} & \lambda + 2 \end{pmatrix} \right] \begin{pmatrix} \Psi_A \\ \Psi_B \end{pmatrix} = 0, \quad (81)$$

which can be diagonalized, since the potential is symmetric and constant. The eigenvalues are

$$\lambda + 3 \pm \sqrt{4\lambda + 9} = \begin{cases} (\ell + 1)(\ell + 2) \\ \ell(\ell - 1) \end{cases}.$$

Having solved Eqs. (80) and (81), the expressions for the one-forms  $A$  and  $B$  in terms of the scalar fields are obtained from the original equations (79):

$$\begin{aligned} A &= -\frac{1}{\lambda} \tilde{*}d \left( \Psi_A + \frac{2}{\sqrt{\lambda + 2}} \Psi_B \right), \\ B &= -\frac{1}{\lambda(\lambda + 2)} \left[ \tilde{*}d \left( 2\Psi_A + \frac{\lambda + 4}{\sqrt{\lambda + 2}} \Psi_B \right) + \lambda d\Pi_B \right]. \end{aligned}$$

Since the operators in Eqs. (80) and (81) are positive, we conclude, using the argument given in Appendix B, that the Schwarzschild solution admits neither unstable nor stationary non-Abelian odd-parity modes with  $\ell > 1$ .

### B. Perturbations with $\ell = 1$

For  $w = 1$  Eq. (58) decouples for all values of  $\ell$ . The vacuum perturbations of the Schwarzschild metric with  $\ell = 1$  are, therefore, governed by Eq. (78) with  $\lambda = 0$ . We have already recalled in Sect. IID that this equation cannot give rise to unstable modes, while it admits the well-behaved stationary solution  $H = (2aM/r)dt$ , giving rise to the Kerr excitation of the Schwarzschild metric,

$$\delta g_{t\varphi} = -a \frac{2M}{r} \sin^2 \vartheta. \quad (82)$$

In order to analyze the YM sector, we first note that the gauge invariant quantities introduced in Eqs. (E4) for  $\ell = 1$  are *not* well-defined if  $w = 1$ . Hence, the  $\ell = 1$  perturbations of the Schwarzschild background require a special treatment: For  $w = 1$  and  $\ell = 1$  we define  $a$ ,  $b$  and  $c$  in the same way as for  $\ell > 1$ , that is, by Eqs. (E6). Hence,  $a = \alpha - d\mu$ ,  $b = \beta$  and  $c = \nu - \mu$ , where the one-forms  $\alpha$ ,  $\beta$  and the scalars  $\mu$ ,  $\nu$  parametrize  $\delta A^{(\ell=1)}$  according to Eq. (28). Since the gauge fields vanish on the background, all YM amplitudes are coordinate-invariant, and it remains to consider their behavior under gauge transformations. By virtue of Eqs. (E1) and (E2)  $c$  remains invariant, whereas  $a$  and  $b$  transform according to  $\rightarrow a + df_2$ ,  $b \rightarrow b + df_2$ . Repeating the arguments given in Sect. IV B, the perturbation equations for  $w = 1$  and  $\ell = 1$  eventually become

$$\begin{aligned} d^\dagger (R^2 da) - 2dc + 4(a - b) &= 0, \\ d^\dagger (R^2 db) + dc - 2(a - b) &= 0, \\ R^2 d^\dagger (a - b) + c &= 0, \end{aligned}$$

where  $c$ ,  $da$ ,  $db$  and  $(a - b)$  are gauge-invariant. Subtracting the first from the second equation, and using the third one to eliminate  $c$ , we obtain an equation for the one-form  $(b - a)$ . This is decoupled in the usual way, that is, by introducing two scalar fields according to

$$c_1 \equiv R^2 d^\dagger (b - a), \quad c_2 \equiv R^2 \tilde{*}d(b - a).$$

Applying the operators  $d^\dagger$  and  $\tilde{*}d$  on the equation for  $(b - a)$  yields the following wave equations for  $c_1$  and  $c_2$ :

$$\left( -\tilde{\Delta} + \frac{2}{R^2} \right) c_1 = 0, \quad \left( -\tilde{\Delta} + \frac{6}{R^2} \right) c_2 = 0. \quad (83)$$

Since the operators are positive, we may use the standard argument to conclude that Eqs. (83) admit neither unstable nor well-behaved stationary modes. Hence,  $c_1 = c_2 = 0$ , implying that  $a = b$  and  $c = 0$ . It therefore remains to solve  $d(R^2 \tilde{*} da) = 0$  for the gauge-invariant scalar field  $\tilde{*} da$ . With respect to Schwarzschild coordinates, the result is  $a = b = (q/r)dt$  plus gauge terms, where  $q$  is a constant of integration. Now using  $\alpha = a + d\mu$ ,  $\beta = b$  and  $\nu = c + \mu$  in Eq. (28) gives  $\delta A^{(\ell=1)} = aX_1 + bX_2 + cYd\tau_r + d(\mu X_1)$ , and thus, with  $c = 0$ ,  $a = b = (q/r)dt$  and  $X_1 + X_2 = \tau_z$ ,

$$\delta A = \tau_z \frac{q}{r} dt$$

plus a pure gauge term. Using  $\delta F = D\delta A = d\delta A$  for  $w = 1$ , this gives rise to the electric field

$$\delta E = -\tau_z \frac{q}{r^2} dr. \quad (84)$$

The solutions (82) and (84) describe the Kerr-Newman excitation of the Schwarzschild metric in first order of the rotation parameter  $a$  and the electric charge  $q$ .

### C. Perturbations with $\ell = 0$

The relevant perturbation equation is Eq. (59) with  $\bar{B} = H = 0$ ,  $w = 1$  and  $\lambda = -2$ . The amplitude  $A = \bar{A}$  is gauge-invariant and, by virtue of Eq. (65), well-defined. Equation (59) becomes  $d^\dagger(R^2 dA) + 2A = 0$ . Using the integrability condition  $d^\dagger A = 0$ , the scalar field  $\Psi$  is defined according to  $\Psi \equiv R^2 \tilde{*} dA$ , in terms of which Eq. (59) becomes

$$\left(-\tilde{\Delta} + \frac{2}{R^2}\right) \Psi = 0, \quad (85)$$

which admits neither unstable nor acceptable stationary solutions. (Note that the RN excitations with  $\ell = 0$  of the Schwarzschild metric lie in the *even* parity sector.)

## VII. STATIONARY PERTURBATIONS OF NON-ABELIAN SOLITONS AND BLACK HOLES

Having analyzed the complete set of non-Abelian odd-parity perturbations (stationary and dynamical) of the Schwarzschild and the RN solutions, we now turn to the general case, that is, to non-Abelian perturbations of static non-Abelian background configurations. The discussion of the corresponding perturbation equations is a considerably more involved task, since the techniques used above cannot be applied if  $w$  is not constant. Our primary goal in this section is to classify all *stationary* odd-parity perturbations of both the BK solitons [1] and the static, spherically symmetric EYM black holes [2].

In the stationary case, the excitations of a spherically symmetric EYM background decouple into two Sturm-Liouville problems, governing the electric and the magnetic perturbations, respectively. The particular case  $\ell = 1$  was analyzed in Refs. [20] and [11] by different means. There we have shown that the electric sector gives rise to a two parameter family of slowly rotating and / or electrically charged black hole excitations, and to a one-parameter family of slowly rotating, electrically charged solitons. In this section we generalize these results as follows: We show that for *all* values of  $\ell \geq 1$  the electric perturbations are governed by a three-channel Sturm-Liouville problem, while the magnetic sector is described by a single Sturm-Liouville equation for  $\ell > 1$  and is trivial for  $\ell = 1$ . A careful analysis then reveals that neither the electric nor the magnetic sector admit well-defined stationary soliton or black hole excitations if  $\ell > 1$ . This establishes the result that the only stationary odd-parity modes of the BK solitons and EYM black holes are the ones found in Ref. [11] for  $\ell = 1$ .

It turns out to be convenient to parametrize the two-dimensional background metric  $\tilde{g}$  in terms of the radial coordinate  $\rho$ , defined such that  $\tilde{g}$  becomes conformally flat,

$$\tilde{g} = -NS^2 dt^2 + \frac{1}{N} dr^2 = \sigma (-dt^2 + d\rho^2), \quad (86)$$

with  $\sigma(\rho) \equiv N(r)S^2(r)$  and  $dr \equiv NSd\rho$ . [The coordinate  $\rho$  generalizes the coordinate  $r_*$  used in the Schwarzschild or the RN case. We also note that  $\tilde{*}dt = -\tilde{*}d\rho$  and  $\sigma\tilde{*}(dt \wedge d\rho) = -1$ .] The invariant one-forms  $A$ ,  $B$  and  $H$  are expanded with respect to  $t$  and  $\rho$ , e.g.,

$$H \equiv H_0 dt + H_1 d\rho. \quad (87)$$

Since we restrict ourselves to stationary perturbations the coefficients  $H_0$ ,  $H_1$ , etc. are functions of  $\rho$  only. As we shall argue below, the equations involving the zero-components, henceforth called electric perturbations, decouple from the equations for the one-components, henceforth called magnetic perturbations.

### A. The electric sector

The electric perturbation equations involve the amplitudes  $H_0$ ,  $A_0$  and  $B_0$  only. [For  $l = 1$ , we may take  $H_0 = h_0$ ,  $A_0 = a_0$  and  $B_0 = b_0$ , since, by virtue of Eqs. (5), (E4) and (E11), these amplitudes are invariant under *stationary* coordinate transformations.] Using the fact that  $[d^\dagger(R^2 dA)]_0 = -\partial_\rho(\sigma^{-1}R^2\partial_\rho A_0)$  for stationary perturbations of a static background, the zero-components of Eqs. (41), (42) and (43) may be cast into the following three-channel Sturm-Liouville equation:

$$\left(-\partial r^2 \partial + K \partial - \partial K^T + L + P\right) \underline{v} = 0, \quad (88)$$

where  $\underline{v} \equiv (H_0/\sqrt{4Gr}, A_0/\zeta, B_0)$ ,  $\zeta \equiv \sqrt{\lambda+2}$ , and where the differential operator  $\partial$  is defined by

$$\partial \equiv \frac{1}{\sigma} \frac{d}{d\rho} = \frac{1}{S} \frac{d}{dr},$$

with  $r$  and  $\rho$  according to Eq. (86). The  $3 \times 3$  matrices  $\mathbf{K}$ ,  $\mathbf{L}$  and  $\mathbf{P}$  are given in terms of the background fields  $w$ ,  $N$  and  $\sigma = S^2 N$ . The only non-vanishing matrix element of  $\mathbf{K}$  is  $\mathbf{K}_{13} = \sqrt{4Gr} \partial w$ , while the symmetric matrices  $\mathbf{L}$  and  $\mathbf{P}$  are

$$\mathbf{L} = \frac{1}{\sigma} \begin{pmatrix} 2N + \lambda & \text{sym.} & \text{sym.} \\ 0 & \lambda + 2(1 + w^2) & \text{sym.} \\ 0 & -2\zeta w & \lambda + (1 + w^2) \end{pmatrix}, \quad (89)$$

and

$$\mathbf{P} = \frac{1}{\sigma} \begin{pmatrix} 4G \frac{(w^2-1)^2}{r^2} + 2G\sigma(\partial w)^2 & \text{sym.} & \text{sym.} \\ \sqrt{4G\zeta} \frac{1-w^2}{r} & 0 & \text{sym.} \\ \sqrt{4G} w \frac{w^2-1}{r} & 0 & 0 \end{pmatrix}. \quad (90)$$

The formally self-adjoint equation (88) holds for all values of  $\ell \geq 1$ . [In particular, for  $\ell = 1$  it is equivalent to the Sturm-Liouville equation derived in [20], which was shown to admit the stationary modes mentioned above [11]. However, the transformation between the two  $\ell = 1$  sets of equations is not algebraic, because the original formulation given in [20] was based on the generalized twist potential.]

Since Eq. (88) has *regular singular* points at the origin,  $r = 0$ , at the horizon,  $r = r_H$  (where  $N(r_H) = 0$ ), and at infinity,  $r = \infty$ , it is possible to compute the number of stationary modes. Applying the standard theory (see, e.g., [21]) we will now discuss the local solution spaces.

### 1. The solution space at the origin

The leading order behavior of the solutions to Eq. (88) in the vicinity of the origin is determined by the centrifugal barrier  $\mathbf{L}$ , as can be seen from the expansions (C1) of the background quantities. The solutions behave like  $r^\alpha$ , where  $\alpha = -(\ell + 2)$ ,  $-(\ell + 1)$ ,  $-\ell$ ,  $\ell - 1$ ,  $\ell$  or  $\ell + 1$ . Hence, the space of regular solutions at  $r = 0$  is *three-dimensional* for all values of  $\ell \geq 1$ . The expansion becomes

$$\begin{aligned} \underline{v}(r) = & d_1 r^{\ell-1} \left[ \underline{e}_- + \frac{\ell-1}{2\ell+1} (2(\ell+2)b+1) b r^2 \underline{e}_- + \mathcal{O}(r^3) \right] \\ & + d_2 r^\ell \left[ \underline{e}_0 + \frac{2b}{2\ell+1} r \underline{e}_- + \mathcal{O}(r^2) \right] \\ & + d_3 r^{\ell+1} [\underline{e}_+ + \mathcal{O}(r)] \end{aligned} \quad (91)$$

where  $\underline{e}_0 = (1, 0, 0)$ ,  $\underline{e}_+ = (0, \zeta, -l)$  and  $\underline{e}_- = (0, \zeta, l+1)$ , and  $d_1$ ,  $d_2$  and  $d_3$  are constants, and where  $b$  is the fixed constant appearing in the expansions (C1) of the background solutions.

### 2. The solution space at infinity

The asymptotic expansions (C2) of the background quantities show that the leading order behavior of the solutions to Eq. (88) is again completely determined by  $\mathbf{L}$ : The solutions behave like  $r^\alpha$ , where again  $\alpha = -(\ell + 2)$ ,  $-(\ell + 1)$ ,  $-\ell$ ,  $\ell - 1$ ,  $\ell$  or  $\ell + 1$ . The space of asymptotically flat solutions is, therefore, *three-dimensional* for  $\ell > 1$ , and *four-dimensional* for  $\ell = 1$ . For  $\ell = 1$  the asymptotic expansion is found to be

$$\begin{aligned} \underline{v}(r) = & \left( c_0 + \frac{c_1}{r} \right) \left[ \underline{e}_- + \mathcal{O}\left(\frac{\log r}{r^2}\right) \right] + \frac{c_2}{r^2} \left[ \underline{e}_0 + \mathcal{O}\left(\frac{1}{r^2}\right) \right] \\ & + \frac{c_3}{r^3} \left[ \left( 1 + (1-\gamma) \frac{2M}{r} \right) \underline{e}_+ + \mathcal{O}\left(\frac{1}{r^2}\right) \right]. \end{aligned} \quad (92)$$

The constant  $c_2$  is proportional to the total angular momentum  $\delta J$ , while  $c_0$  and  $c_1$  are proportional to the asymptotic value of the electric YM potential  $\delta\Phi_\infty$  and the electric YM charge  $\delta Q_e$ , respectively: Using the above expansion in the expressions (F1) for the linearized local Komar integrals, we find [with  $\tilde{*}F_B = \sigma^{-1}(B'_0 + w'H_0/R^2)$  etc.],

$$\delta Q_e(r \rightarrow \infty) \sim \underline{e}_m \cdot \underline{\mathcal{I}} c_1, \quad \delta J(r \rightarrow \infty) \sim \delta_{m0} c_2.$$

Furthermore, the above expansion, together with the definition (53) and  $\delta\Phi = \delta A(\partial_t)$ , shows that  $\delta\Phi_\infty$  is proportional to  $c_0$ . It is worthwhile recalling that, in contrast to the Abelian case,  $c_0$  cannot be “gauged away”. This is also obvious from the fact that the expression for  $\delta F$  involves an asymptotically vanishing term proportional to  $c_0/r$ , unless for  $w = 1$ .

### 3. The solution space at the horizon

Using the background expansions (C3) at the horizon, the solutions to Eq. (88) behave like  $(r - r_H)^\alpha$ , where the eigenvalues are  $\alpha = 0$  and  $\alpha = 1$ , and the multiplicity is three in both cases. For  $\alpha = 0$  the three eigenvectors may pick up logarithmic terms in next-to-leading order, which destroy the regularity of the horizon. A careful analysis shows that the number of eigenvectors with logarithmic terms in next-to-leading order is equal to the rank of the symmetric matrix

$$\mathbf{S}_1 = \begin{pmatrix} \lambda + 4G \frac{(w_H^2-1)^2}{r_H^2} & \text{sym.} & \text{sym.} \\ \sqrt{4G\zeta} \frac{1-w_H^2}{r_H} & \lambda + 2(1 + w_H^2) & \text{sym.} \\ \sqrt{4G} w_H \frac{w_H^2-1}{r_H} & -2\zeta w_H & \lambda + 1 + w_H^2 \end{pmatrix},$$

which is proportional to the leading order term of  $\mathbf{L} + \mathbf{P}$  in  $r - r_H$ . The determinant of  $\mathbf{S}_1$  is given by

$$\det \mathbf{S}_1 = \lambda [\lambda^2 + (3 - w_H^2)\lambda + 2(1 - w_H^2)^2 + 8G G_H^2],$$

where we recall that  $w_H \equiv w(r_H)$  and  $G_H \equiv w_H(w_H^2 - 1)/r_H$ . This shows that the rank of  $\mathbf{S}_1$  is three for  $\ell > 1$ ,

while one may also verify that the rank is two for  $\ell = 1$ . Hence, all solutions with  $\alpha = 0$  must be excluded, unless  $\ell = 1$ , in which case there exists one acceptable eigenvector. The physical space of solutions at  $r = r_H$  is, therefore, *three-dimensional* for  $\ell > 1$  and *four-dimensional* for  $\ell = 1$ .

#### 4. Soliton excitations

Since the BK background is continuous, and since the perturbation equations are linear with continuous coefficients for  $0 < r < \infty$ , the local solutions (91) and (92) admit extensions to the semi-open intervals  $[0, \infty)$  and  $(0, \infty]$ , respectively. Since, for  $\ell = 1$ , these solution subspaces are three- and four-dimensional, respectively, and since the total space of solutions is *six-dimensional*, we conclude that the intersection space is generically one-dimensional. Hence, there exists (at least) one global solution, describing the rotating charged solitons found in [11].

For  $\ell > 1$ , the intersection space is generically trivial, since the solution spaces are three-dimensional at both the origin *and* infinity. Hence, there exist no generic soliton excitations for  $\ell > 1$ . In fact, *non-generic* solutions are excluded as well, as we shall prove below.

#### 5. Black hole excitations

Applying the same argument as in the soliton case, we conclude that Eq. (88) admits a *two-dimensional* intersection space of global solutions for  $\ell = 1$ , since the local solution spaces at the horizon and at infinity are four-dimensional. The solutions give rise to the black hole excitations found in [11], which are parametrized by their total angular momentum  $\delta J$  and their electric YM charge  $\delta Q_e$ .

For  $\ell > 1$  there exist again no generic solutions, since the solution spaces at the horizon and at infinity are three-dimensional only. It therefore remains to exclude non-generic solutions, which we shall do next.

#### 6. Absence of non-generic solutions for $\ell > 1$

Our aim is to show that Eq. (88) with the boundary conditions discussed above admits neither soliton nor black hole solutions for  $\ell > 1$ . We do so by casting Eq. (88) into the form required to apply the argument outlined in Appendix B. This is achieved by performing the linear transformation  $\underline{v} = \underline{T}\underline{u}$ , which yields

$$(-\partial \underline{A} \partial + \underline{S}) \underline{u} = 0, \quad (93)$$

where  $\underline{A}$  is symmetric and positive, while  $\underline{S}$  is symmetric and positive semidefinite. The linear transformation  $\underline{T}$  is given by  $\underline{T} = \underline{T}_1 \circ \underline{T}_2$ , where

$$\underline{T}_1 = \text{diag}(r, 1, 1), \quad \underline{T}_2 = \mathbb{1} - \sqrt{4G} \begin{pmatrix} 0 & 0 & 0 \\ 1/\zeta & 0 & 0 \\ w & 0 & 0 \end{pmatrix}$$

[Note that the components of  $\underline{u} = \underline{T}^{-1}\underline{v}$  coincide with the amplitudes introduced in Eq. (47):  $\sqrt{4G}u_1 = H_0/r^2$ ,  $\zeta u_2 = A_0 + H_0/r^2$ ,  $u_3 = B_0 + wH_0/r^2$ .] The Sturm-Liouville equation (88) now assumes the desired form (93), with the symmetric matrices  $\underline{A} = r^2 \underline{T}^T \circ \underline{T}$ ,  $\underline{S} = \underline{T}_2^T \circ \tilde{\underline{S}} \circ \underline{T}_2$ , where

$$\tilde{\underline{S}} = \frac{1}{\sigma} \begin{pmatrix} \lambda r^2 + 4G(w^2 - 1)^2 & \text{sym.} & \text{sym.} \\ \sqrt{4G}\zeta(1 - w^2) & \lambda + 2(1 + w^2) & \text{sym.} \\ \sqrt{4G}w(w^2 - 1) & -2\zeta w & \lambda + 1 + w^2 \end{pmatrix}.$$

It is not hard to see that the matrix  $\tilde{\underline{S}}$  is positive for all values of  $\ell > 1$  and positive semidefinite for  $\ell = 1$ . Furthermore, by virtue of the expansions given above for  $\ell > 1$ , the boundary term  $\underline{u} \cdot \underline{A} \partial \underline{u}$  vanishes at the origin, at the horizon, and at infinity. Both soliton and black hole solutions are, therefore, excluded as a consequence of the argument given in Appendix B.

We emphasize that the boundary terms at the origin and at the horizon do give non-vanishing contributions if  $\ell = 1$ . The positive operator in Eq. (93) is, therefore, self-adjoint only for  $\ell > 1$ .

#### 7. Conclusion

We have proven the following *local uniqueness theorems* for odd-parity perturbations in the electric sector: The only stationary, asymptotically flat *black hole* solutions which are infinitesimally close to the static, spherically symmetric EYM black holes are the rotating and/or electrically charged excitations in the  $\ell = 1$  sector. The only *soliton* solutions which are infinitesimally close to the BK solitons are the electrically charged excitations in the  $\ell = 1$  sector.

These results are in agreement with the non-Abelian staticity theorem [22], which asserts that spacetime is static and purely magnetic if the combination  $\Omega_H J - \text{Tr} \{ \Phi_\infty Q_e \}$  vanishes, where  $\Omega_H$  is the angular velocity of the horizon: For  $\ell = 1$ , non-static solitons and black holes can exist, while, for  $\ell > 1$ , there is no contribution to  $J$  and  $Q_e$  [see the general formulae (F1)], implying that the non-static and electric contributions  $H_0$ ,  $A_0$  and  $B_0$  must vanish.

#### B. The magnetic sector

For stationary perturbations one has  $d^\dagger A = -\sigma^{-1} A'_1$ , where here and in the following a prime denotes differentiation with respect to the radial coordinate  $\rho$ , defined in Eq. (86). Since the background is static, one also

has  $\langle A, dw \rangle = -\sigma^{-1} w' A_1$ . Hence, the gravitational constraint (40) and the YM constraints (45), (46), as well as Eq. (44) involve only the one-components of  $A$ ,  $B$  and  $H$ . It is, therefore, possible to express the YM amplitudes  $A_1$ ,  $B_1$  and  $C$  in terms of the gravitational perturbation  $H_1$ :

$$\begin{aligned} A_1 &= \left( w^2 - 1 + \frac{R^2}{2G} \right) \frac{H_1}{R^2}, \\ B_1 &= \frac{1}{4Gw'} H_1', \\ C &= -\frac{1}{\sigma} \left[ w' H_1 + \frac{R^2}{4G} \left( \frac{H_1'}{w'} \right)' \right]. \end{aligned} \quad (94)$$

Using the above expressions and the circumstance that  $[d^\dagger(R^4 d(H/R^2))]_1$  vanishes for stationary perturbations of a static background, the one-component of the gravitational equation (41) yields the following Sturm-Liouville equation for  $H_1$ :

$$\left[ -\frac{d}{d\rho} \frac{1}{w'^2} \frac{d}{d\rho} + \frac{[\ell(\ell+1) - 2w^2]\sigma - 4Gw'^2}{R^2 w'^2} \right] H_1 = 0, \quad (95)$$

where we recall that  $w' = dw/d\rho = NSdw/dr$ .

The above equation holds for  $\ell > 1$  only. For  $\ell = 1$  the perturbations are governed by Eqs. (56). Since the one-components of the first terms in these equations vanish for stationary perturbations, we obtain  $\bar{a}_1 = \bar{b}_1 = 0$ , provided that  $w^2 - 1$  does not vanish everywhere. Now using the fact that there exists a gauge for which  $h_\rho$  vanishes if  $\ell = 1$ , we conclude that magnetic excitations cannot exist for  $\ell = 1$ . (The case  $w^2 = 1$ ,  $\ell = 1$  has already been discussed in Sect. VIB.)

Equation (95) has regular singular points at the origin,  $R = 0$ , at the horizon,  $N = \langle dR, dR \rangle = 0$ , at infinity,  $R = \infty$ , and at all points where  $w'$  vanishes. (For the one-node background solutions this is only the case at the origin and at infinity.) In order to conclude that Eq. (95) generically admits neither acceptable soliton nor black hole excitations, it is sufficient to discuss the regular singular points at the boundaries in leading order.

### 1. Soliton excitations

Using the expansions (C1) for the BK background at the origin shows that the fundamental solutions to Eq. (95) behave like  $r^{\ell+2}$  and  $r^{1-\ell}$ . Since  $\ell > 1$ , the subspace of solutions giving rise to finite metric perturbations is, therefore, *one-dimensional* at the origin. In the asymptotic region one uses the expansions (C2) to conclude that the fundamental solutions behave like  $r^{-\ell-2}$  and  $r^{\ell-1}$ , implying that the subspace of bounded solutions is again *one-dimensional*. Generic soliton excitations are, therefore, excluded. [The subspace of bounded solutions at the inner points  $w' = 0$  turn out to be two-dimensional. It is, however, generically not possible to match the solutions from  $r = 0$  and  $r = \infty$  at the points  $w' = 0$  such that the amplitude  $C$  is continuous.]

### 2. Black hole excitations

Using the horizon expansions (C3) shows that the fundamental solutions to Eq. (95) behave like  $(r - r_H)^0$  and  $(r - r_H)^2$ . The first solution is physically unacceptable, since the invariant quantity  $\langle H, H \rangle = H_1^2/\sigma$  diverges for  $r \rightarrow r_H$ . Hence, the physical subspaces at the horizon and at infinity are *one-dimensional*, implying that black hole excitations do not exist in the generic case.

So far, we were not able to exclude non-generic solutions by rigorous means: The first problem is that the potential in Eq. (95) is not manifestly positive (although numerical investigation suggest that this is the case). Furthermore, the boundary term arising in the integral argument given in Appendix B does not vanish at points where  $w' = 0$ . It is, however, clear that the potential is positive if  $\ell$  is big enough. In this case the integral argument applies, at least for excitations of the background solutions with one node.

### 3. Conclusion

Since  $H_1$  parametrizes the *non-circular* part of the metric, we have shown that there exists no non-circular deformations in the odd-parity sector. This completes the classification of the stationary odd-parity excitations of the BK solitons and the corresponding non-Abelian black holes. The only physically admissible non-Abelian stationary odd-parity excitations of these configurations are the rotating, electrically charged solitons and the two-parameter family of black holes found in [11]. All modes lie in the electric part of the distinguished sector  $\ell = 1$ .

## VIII. DYNAMICAL PERTURBATIONS

Stationary perturbations need to be analyzed in order to find equilibrium solutions which are infinitesimally neighbored to known static configurations, or to establish local uniqueness results. The linear *stability* properties of static background solutions are, however, described by *non-stationary* perturbations. In order to study their dynamical behavior by means of spectral theory, it is necessary to cast the perturbation equations into a system of *pulsation* equations, that is, into a wave equation whose spatial part is (formally) *self-adjoint*. Using the static EYM soliton or black hole background, our task is, therefore, to write the perturbation equations (58) - (61) in the form

$$\left[ \frac{\partial^2}{\partial t^2} + \mathcal{A} \right] u = 0, \quad (96)$$

where  $\mathcal{A}$  is a self-adjoint operator, containing spatial derivatives up to second order. For perturbations of the Schwarzschild and RN black holes this was achieved in



Eqs. (69), (71), (80) and (81). For perturbations of non-Abelian background configurations, however, one needs to proceed differently:

For  $\ell = 0$  (i.e., for radial perturbations), the above task was achieved in [15], where it was shown that the static, spherically symmetric BK solitons and EYM black holes have exactly  $n$  unstable radial modes in the odd-parity sector,  $n$  being the number of nodes of  $w$ .

For  $\ell = 1$ , we will show below that the metric perturbations decouple, and that the perturbation equations can be cast into a wave equation for the remaining YM perturbations, where the operator  $\mathcal{A}$  is symmetric and positive. This will establish the absence of unstable odd-parity modes in the sector  $\ell = 1$ .

For  $\ell > 1$ , we were not able to derive symmetric equations in terms of the gauge-invariant amplitudes  $H$ ,  $A$ ,  $B$  and  $C$ . However, a system of hyperbolic equations can be obtained as follows: By virtue of Eqs. (40), (44) and (45) one can express the time derivatives of the electric components  $H_0$ ,  $A_0$  and  $B_0$  in terms of the magnetic components  $H_1$ ,  $A_1$ ,  $B_1$  and  $C$  and their first spacial derivatives. Equations (41), (42), (43) and (46) then yield a hyperbolic system of the form

$$\left[ \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial \rho^2} + \mathbf{K} \frac{\partial}{\partial \rho} + \mathbf{V} \right] u = 0,$$

where  $u$  comprises the magnetic components,  $u = (H_1, A_1, B_1, C)$ , and where the radial coordinate  $\rho$  is defined as in (86). Unfortunately, neither the first order derivatives nor the potential  $\mathbf{V}$  are formally self-adjoint.

In [14] we have argued that the gauge-invariant amplitudes used in the present paper are not suited to describe dynamical perturbations, an exception being vacuum gravity or self-gravitating Abelian fields. In order to obtain a symmetric wave equation one needs to introduce amplitudes which are adapted to the *staticity* rather than the spherical symmetry of the background. In terms of these new, *curvature-based* amplitudes, the odd-parity pulsation equations can be cast into the desired form (96), as we have shown in [14].

In the remainder of this section we present the distinguished cases  $\ell = 0$  and  $\ell = 1$ . In these situations the perturbation equations can be written in the desired form, since gravity can be decoupled for  $\ell = 1$ , while for  $\ell = 0$  only YM perturbations are present in the odd-parity sector.

### A. The pulsation equation for $\ell = 0$

For spherically symmetric perturbations the pulsation equation is obtained from Eq. (57),

$$d^\dagger (R^2 da) + 2w^2 a = 0,$$

for the gauge-invariant YM amplitude  $a = \alpha - d(\nu/w)$ ; see Eq. (E8). This equation is not regular at points where

$w$  vanishes. (The case where  $w$  vanishes identically was discussed in Sect. VC. Introducing the regular one-form  $w^2 a = w^2 \alpha + \nu dw - w d\nu$ , and defining the potential  $\Phi$  by the equation  $w^2 a = \tilde{*}d(w\Phi)$ , we find

$$-\tilde{\Delta}\Phi + \left[ 2\left\langle \frac{dw}{w}, \frac{dw}{w} \right\rangle + \frac{1}{R^2}(w^2 + 1) \right] \Phi = 0,$$

where we have also used the background YM equation (21). For a static background we may assume a time dependence of the form  $\exp(i\omega t)$ , which yields

$$\left[ -\frac{\partial^2}{\partial \rho^2} + 2\frac{w'^2}{w^2} + \frac{\sigma}{R^2}(w^2 + 1) \right] \Phi = \omega^2 \Phi, \quad (97)$$

where a prime denotes the differentiation with respect to  $\rho$ , and now  $\Phi \equiv \Phi(\rho)$ . In order to overcome the difficulty that the potential is singular at points where  $w$  vanishes, one may perform the following super-symmetric transformation: First, the operator on the LHS can be factorized and written as  $Q^\dagger Q$ , with  $Q$  and  $Q^\dagger$  according to

$$Q = \frac{1}{w} \frac{\partial}{\partial \rho} w + u, \quad Q^\dagger = -w \frac{\partial}{\partial \rho} \frac{1}{w} + u,$$

where  $u$  is subject to the differential equation

$$-w^2 \left( \frac{u}{w^2} \right)' + u^2 = \frac{2\sigma w^2}{R^2}. \quad (98)$$

One may then write Eq. (97),  $Q^\dagger Q \Phi = \omega^2 \Phi$ , in terms of  $\Psi \equiv Q\Phi$ , which yields  $Q Q^\dagger \Psi = \omega^2 \Psi$ . Since  $\omega^2 \Phi = Q^\dagger \Psi$ , there is a one-to-one correspondence between  $\Phi$  and  $\Psi$ , provided that  $\omega \neq 0$ . Furthermore,  $\Psi$  is normalizable if  $\Phi$  is normalizable, and vice-versa, since  $\langle \Psi, \Psi \rangle = \langle Q\Phi, Q\Phi \rangle = \langle Q^\dagger Q\Phi, \Phi \rangle = \omega^2 \langle \Phi, \Phi \rangle$ .

The equivalent problem,  $Q Q^\dagger \Psi = \omega^2 \Psi$ , reads

$$\left[ -\frac{\partial^2}{\partial \rho^2} + \frac{\sigma}{R^2}(3w^2 - 1) + 2u' \right] \Psi = \omega^2 \Psi, \quad (99)$$

where now the potential is regular, provided that  $u$  is a regular solution of Eq. (98). Since the function

$$\Psi_0 = w \exp \int_{\rho_0}^{\rho} u(\tilde{\rho}) d\tilde{\rho}$$

satisfies  $Q^\dagger \Psi_0 = 0$ , it is a solution to Eq. (99) for  $\omega = 0$ . The key observation in [15] is that there exists a solution to Eq. (98) such that  $u/w^2$  and  $u'$  are regular and  $\Psi_0$  is normalizable. Since the factor  $w$  causes  $\Psi_0$  to have exactly  $n$  nodes ( $n$  being the number of nodes of  $w$ ), this establishes the fact that the transformed pulsation equation (99) admits exactly  $n$  unstable modes.

It remains to show that each unstable mode of Eq. (99) can be realized by a regular choice of the original amplitudes  $\alpha$  and  $\nu$ . In order to see this, we first note that for  $\omega \neq 0$  the inverse transformation becomes

$$w\Phi = \frac{1}{\omega^2} (-w\Psi' + w'\Psi + wu\Psi),$$

implying that the gauge-invariant combination  $w^2a$  is regular. Finally, one adopts the *temporal gauge*,  $\alpha_t = 0$ , with respect to which Eq. (97) yields

$$\frac{\partial}{\partial t}\alpha_\rho = \frac{2\sigma}{R^2}w\Phi,$$

implying that  $\alpha_\rho$  is regular. Using  $\Psi = Q\Phi = w^{-1}(w\Phi)' + u\Phi$ , as well as the  $t$ -component of  $w^2a = \tilde{*}d(w\Phi)$  in the temporal gauge, gives

$$\frac{\partial}{\partial t}\nu = \Psi - \frac{u}{w}w\Phi.$$

This establishes the existence of exactly  $n$  unstable modes of the original perturbation equations, since  $u/w$  can be chosen to be regular, implying that  $\nu$  is regular.

### B. The pulsation equation for $\ell = 1$

We now show that for  $\ell = 1$  the gravitational perturbations can be expressed in terms of the YM perturbations, which yields a pulsation equation for the YM amplitudes. The gravitational amplitude  $h$  enters the perturbation equations (56) only via the coordinate-invariant combination  $F_h = d(R^{-2}h)$ . The crucial observation is that the second plus  $2w$  times the third minus  $2G$  times the first equation in (56) yields the conservation law

$$d^\dagger \left[ -\frac{1}{2G}R^4F_h + R^2(F_a + 2wF_b) \right] = 0.$$

Recalling the definitions  $F_a = d\bar{a} - F_h$  and  $F_b = d\bar{b} - wF_h$ , we find after integrating the above equation

$$F_h = f \left( d\bar{a} + 2w d\bar{b} + \frac{c_0}{R^2}\tilde{*}1 \right),$$

where  $c_0$  is a constant, and  $f$  denotes the background quantity  $f \equiv (R^2/2G + 1 + 2w^2)^{-1}$ . Using this expression for  $F_h$  in Eqs. (56) yields the symmetric, inhomogeneous equation

$$d^\dagger \left[ \mathbf{G} d \begin{pmatrix} \bar{a} \\ \bar{b} \end{pmatrix} \right] + \mathbf{F} \begin{pmatrix} \bar{a} \\ \bar{b} \end{pmatrix} = -c_0\tilde{*}d \begin{pmatrix} f \\ 2wf \end{pmatrix} \quad (100)$$

for the gauge- and coordinate-invariant YM amplitudes  $\bar{a}$  and  $\bar{b}$ . The  $2 \times 2$  matrices  $\mathbf{G}$  and  $\mathbf{F}$  are symmetric and given in terms of the background quantities by

$$\mathbf{G} = \frac{fR^2}{2} \begin{pmatrix} 4w^2 + R^2/G & -4w \\ -4w & 4 + 2R^2/G \end{pmatrix},$$

$$\mathbf{F} = 2 \begin{pmatrix} 1 + w^2 & -2w \\ -2w & 1 + w^2 \end{pmatrix},$$

where  $\mathbf{F}$  is positive definite for  $w^2 \neq 1$ . (The case  $w \equiv 1$  was already discussed in Sect. VIB.)

The one-forms  $\bar{a}$  and  $\bar{b}$  may be expanded with respect to Schwarzschild coordinates  $t$  and  $\rho$ ,

$$\begin{pmatrix} \bar{a} \\ \bar{b} \end{pmatrix} = E dt + B d\rho,$$

where  $E$  and  $B$  represent the gauge-invariant electric and magnetic YM fields, respectively. Using this in Eq. (100) gives, for a static background,

$$-\frac{\partial}{\partial \rho} \left[ \frac{1}{\sigma} \mathbf{G} (E' - \dot{B}) \right] + \mathbf{F} E = c_0 \frac{\partial}{\partial \rho} \begin{pmatrix} f \\ 2wf \end{pmatrix},$$

$$-\frac{\partial}{\partial t} \left[ \frac{1}{\sigma} \mathbf{G} (E' - \dot{B}) \right] + \mathbf{F} B = 0,$$

where the dot and the prime denote differentiations with respect to  $t$  and  $\rho$ , respectively. In particular, for stationary perturbations,  $\dot{E} = \dot{B} = 0$ , we recover the facts that the electric and the magnetic perturbations decouple, and that  $B$  vanishes.

For dynamical perturbations a homogeneous pulsation equation of the desired form is obtained as follows: Differentiating the first equation with respect to  $t$  and the second one with respect to  $\rho$  yields the relation

$$\mathbf{F} \dot{E} = (\mathbf{F} B)',$$

where we have also taken advantage of the fact that the background is static. Using this to eliminate  $\dot{E}$  from the second equation, we obtain the following two-channel wave equation with formally self-adjoint spatial part:

$$\left[ \frac{\partial^2}{\partial t^2} - \mathbf{Q} \frac{\partial}{\partial \rho} \mathbf{Q}^{-2} \frac{\partial}{\partial \rho} \mathbf{Q} + \frac{\sigma}{R^2} \begin{pmatrix} w^2 + 2 & -3w \\ -3w & 2w^2 + 1 \end{pmatrix} \right. \\ \left. + \frac{4G}{R^4} \sigma (1 - w^2)^2 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right] \mathbf{Q} B = 0, \quad (101)$$

where  $\mathbf{Q}$  satisfies  $\mathbf{F} = 2\mathbf{Q}^2$ ,

$$\mathbf{Q} = \begin{pmatrix} 1 & -w \\ -w & 1 \end{pmatrix}.$$

Since the operator is symmetric and positive, we conclude that the spherically symmetric EYM solitons and black holes have no unstable odd-parity excitations in the sector  $\ell = 1$ .

## APPENDIX A: LINEARIZED RICCI AND EINSTEIN TENSORS

In this Appendix we give the expressions for the linearized Christoffel symbols and the Ricci and Einstein tensors. As we have argued in Sect. IIC, it is sufficient to compute the perturbations in the ODG. The Christoffel symbols for an arbitrary (not necessarily static) spherically symmetric spacetime are

$$\Gamma_{bc}^A = 0, \quad \Gamma_{Bc}^a = 0,$$

$$\Gamma_{BC}^A = \hat{\Gamma}_{BC}^A, \quad \Gamma_{bc}^a = \tilde{\Gamma}_{bc}^a,$$

$$\Gamma_{Bc}^A = \delta_B^A R^{-1} \tilde{\nabla}_c R, \quad \Gamma_{BC}^a = -\hat{g}_{BC} R \tilde{\nabla}^a R,$$

where  $\tilde{\nabla}$  denotes the covariant derivative operator with respect to the two-dimensional metric  $\tilde{g}$  defined in Eq. (1). In the ODG the metric perturbations (3) and their inverse become

$$\begin{aligned}\delta g_{ab} &= \delta g^{ab} = \delta g_{AB} = \delta g^{AB} = 0, \\ \delta g_{Ab} &= h_b S_A, \quad \delta g^{Ab} = -h^b S^A,\end{aligned}$$

where all indices are raised with the background metric, i.e.,  $h^b \equiv \tilde{g}^{ab} h_a$  and  $S^A \equiv g^{AB} S_B \equiv R^{-2} \hat{g}^{AB} S_B$ . Using this, and the background metric (1), the perturbed Christoffel symbols,  $\delta \Gamma_{\alpha\beta}^\mu = \frac{1}{2} g^{\mu\nu} (\delta g_{\alpha\nu;\beta} + \delta g_{\beta\nu;\alpha} - \delta g_{\alpha\beta;\nu})$ , become in the ODG

$$\begin{aligned}\delta \Gamma_{bc}^a &= 0, \quad \delta \Gamma_{BC}^A = S^A \hat{g}_{BC} R h^a \tilde{\nabla}_a R, \\ \delta \Gamma_{BC}^a &= h^a \hat{\nabla}_{\{B} S_{C\}}, \quad \delta \Gamma_{bc}^A = S^A \hat{\nabla}_{\{b} h_{c\}},\end{aligned}$$

and

$$\begin{aligned}\delta \Gamma_{Bc}^a &= S_B \tilde{g}^{ad} (\tilde{\nabla}_{[c} h_{d]} - h_d R^{-1} \tilde{\nabla}_c R), \\ \delta \Gamma_{Bc}^A &= h_c R^{-2} \hat{g}^{AD} \hat{\nabla}_{[B} S_{D]},\end{aligned}$$

where  $\tilde{\nabla}_{[b} h_{a]} \equiv \frac{1}{2} (\tilde{\nabla}_b h_a - \tilde{\nabla}_a h_b)$  and  $\tilde{\nabla}_{\{b} h_{a\}} \equiv \frac{1}{2} (\tilde{\nabla}_b h_a + \tilde{\nabla}_a h_b)$ . It is now a straightforward task to compute the perturbed Ricci tensor in the ODG. Using  $\delta R_{\alpha\beta} = \delta \Gamma_{\alpha\beta;\mu}^\mu - \delta \Gamma_{\alpha\mu;\beta}^\mu$ , one finds

$$\delta R_{ab} = 0, \quad \delta R_{AB} = \tilde{\nabla}^a h_a \hat{\nabla}_{\{A} S_{B\}}, \quad (A1)$$

and

$$\begin{aligned}\delta R_{Ab} &= \frac{S_A}{R^2} \tilde{\nabla}^a \left[ R^4 \tilde{\nabla}_{[b} (h_{a]} R^{-2}) \right] + \frac{h_b}{R^2} \hat{g}^{BC} \hat{\nabla}_C \hat{\nabla}_{[A} S_{B]} \\ &\quad - \frac{S_A}{R^2} \left( R \tilde{\Delta} R + \langle dR, dR \rangle \right) h_b.\end{aligned} \quad (A2)$$

In order to simplify the expression for  $\delta R_{Ab}$  we take advantage of the background equations (2) to write  $R \tilde{\Delta} R + \langle dR, dR \rangle = 1 - \frac{1}{2} \hat{g}^{AB} R_{AB}$ . Also using the transversality of the spherical vector harmonics,  $\hat{g}^{AB} \hat{\nabla}_A S_B = 0$ , we find

$$2 \hat{g}^{BC} \hat{\nabla}_C \hat{\nabla}_{[A} S_{B]} = - \left( \hat{\Delta} S \right)_A = \ell(\ell+1) S_A,$$

which we apply in the second term of Eq. (A2). Finally using the fact that  $\delta G_{Ab} = \delta R_{Ab} - \frac{1}{2} S_A h_b g^{\mu\nu} R_{\mu\nu}$  in the ODG, we obtain (with the identity  $R^2 \tilde{g}^{ab} R_{ab} = \hat{g}^{AB} G_{AB} = R^2 G_B^B$ ) the result

$$\delta G_{Ab} = \frac{S_A}{R^2} \left\{ \tilde{\nabla}^a \left[ R^4 \tilde{\nabla}_{[b} (h_{a]} R^{-2}) \right] + \frac{h_b}{2} (\lambda + R^2 G_B^B) \right\}, \quad (A3)$$

which holds in the ODG. The expression for  $\delta G_{AB}$  follows from the fact that in the ODG  $\delta G_{AB} = \delta R_{AB}$ ,

$$\delta G_{AB} = \hat{\nabla}_{\{A} S_{B\}} \tilde{\nabla}^b h_b. \quad (A4)$$

Eventually,  $\delta G_{ab} = \delta R_{ab} - \frac{1}{2} R^{-2} \tilde{g}_{ab} \hat{g}^{AB} \delta R_{AB}$  in the ODG, which vanishes by virtue of Eqs. (A1) and the transversality of the spherical vector harmonics.

$$\delta G_{ab} = 0. \quad (A5)$$

The ODG expressions (A3)-(A5) together with Eqs. (8) and (9) evaluated in the ODG yield the desired formulae (11).

## APPENDIX B: STATIONARY SOLUTIONS OF RW-TYPE EQUATIONS

We discuss the conditions under which the stationary RW type differential equation,

$$\left[ -\partial_r N(r) \partial_r + \frac{1}{r^2} V(r) \right] \Psi = 0, \quad (B1)$$

admits only the trivial solution. The potential  $V(r)$  and the function  $N(r)$  are assumed to be non-negative for  $r_H < r < \infty$ , and to have analytical expansions of the form

$$N(r) = N_1(r-r_H) + \mathcal{O}(r-r_H)^2, \quad V(r) = V_H + \mathcal{O}(r-r_H)$$

in the vicinity of the horizon  $r_H$ , and

$$N(r) = 1 + \mathcal{O}(r^{-1}), \quad V(r) = \ell(\ell+1) + \mathcal{O}(r^{-1})$$

as  $r \rightarrow \infty$ , where  $\ell > 0$ ,  $N_1 \neq 0$ ,  $V \neq 0$ . [In particular, the RW equation (17) meets the above conditions, and so does the Zerilli equation, describing vacuum perturbations with even parity.] Under the above conditions the differential equation (B1) has regular singular points [21] at  $r = r_H$  and  $r = \infty$ , implying that  $\Psi$  behaves like

$$\Psi = \begin{cases} P_1(r-r_H) \\ \log(r-r_H) Q_1(r-r_H) \end{cases} \quad \text{for } r \rightarrow r_H,$$

and

$$\Psi = \begin{cases} r^{-\ell-1} P_2(r^{-1}) \\ r^\ell Q_2(r^{-1}) \end{cases} \quad \text{for } r \rightarrow \infty.$$

Here, the  $P_{1,2}(x)$  and  $Q_{1,2}(x)$  are locally convergent power series with  $P_{1,2}(0) \neq 0$  and  $Q_{1,2}(0) \neq 0$ .

For non-negative  $N(r)$  and  $V(r)$  the standard integral argument,

$$\begin{aligned}0 &\leq \int_{r_1}^{r_2} \left( N(\partial_r \Psi)^2 + \frac{1}{r^2} V \Psi^2 \right) dr \\ &= \int_{r_1}^{r_2} \left( -\partial_r N \partial_r \Psi + \frac{1}{r^2} V \Psi \right) \Psi dr + [N \Psi \partial_r \Psi]_{r_1}^{r_2},\end{aligned}$$

implies that Eq. (B1) has only the trivial solution  $\Psi = 0$ , provided that the boundary term vanishes in the limit  $r_1 \rightarrow r_H$  and  $r_2 \rightarrow \infty$ . In particular, this is the case if

the asymptotic flatness and regularity conditions imply that the solutions with  $Q_1$  and  $Q_2$  must be excluded.

As an example, stationary solutions of the RW equation (17) with  $\ell \geq 2$  can be excluded as follows: The variation of the curvature components  $\delta R_{ABcd}^{inv} dx^c \wedge dx^d = 2R^2 d(H/R^2) \hat{\nabla}_{[B} S_{A]}$  must be bounded, implying that  $\Psi/R$  must remain bounded as well. Hence, the solution with  $Q_1$  is not admissible, and neither is the one with  $Q_2$ , unless  $\ell = 1$ .

### APPENDIX C: EINSTEIN-YANG-MILLS BACKGROUND SOLUTIONS

In this Appendix we recall the behavior of the static, spherically symmetric soliton and black hole solutions to the EYM equations (24)-(26) at the singular points (see, e.g., [23]): In the vicinity of the origin one has (with  $G = 1$ )

$$\begin{aligned} N(r) &= 1 - 4b^2 r^2 + O(r^4), \\ S(r) &= S_0 [1 + 4b^2 r^2 + O(r^4)], \\ w(r) &= 1 - br^2 + O(r^4), \end{aligned} \quad (C1)$$

with parameters  $b = -\frac{1}{2}w''(0)$  and  $S_0 > 0$ . In the asymptotic regime one finds

$$\begin{aligned} N(r) &= 1 - 2Mr^{-1} + O(r^{-2}), \\ S(r) &= 1 + O(r^{-4}), \\ w(r) &= \pm [1 - \gamma 2Mr^{-1} + O(r^{-2})], \end{aligned} \quad (C2)$$

with parameters  $M$  and  $\gamma$ . Finally, in the vicinity of the horizon the behavior is given by

$$\begin{aligned} N(r) &= \frac{F_H}{r_H} (r - r_H) + O(r - r_H)^2, \\ S(r) &= S_H \left[ 1 + \frac{G_H^2}{r_H F_H^2} (r - r_H) + O(r - r_H)^2 \right], \\ w(r) &= w_H + \frac{G_H}{F_H} (r - r_H) + O(r - r_H)^2, \end{aligned} \quad (C3)$$

where  $F_H = 1 - (w_H^2 - 1)^2 / r_H^2$  and  $G_H = w_H(w_H^2 - 1) / r_H$ . Here, the free parameters are  $w_H \equiv w(r_H)$  and  $S_H$ .

### APPENDIX D: SU(2)-VALUED HARMONIC ONE-FORMS

We construct a basis of su(2)-valued spherical harmonic one-forms which transform canonically under the angular momentum operator  $J$ , defined by

$$J_X \mathbf{T} \equiv \mathcal{L}_X \mathbf{T},$$

where  $\mathbf{T}$  is a tensor field over the *spherically symmetric* (pseudo-)Riemannian manifold  $(M, \mathbf{g})$ . Here  $\mathcal{L}_X$  denotes the Lie derivative with respect to an infinitesimal rotation

$X$  on  $M$ . [In particular, for infinitesimal rotations in  $\mathbb{R}^3$  about the  $x^k$ -axis, we define  $J_k \equiv J_{X_k}$ , where  $(X_k)_{rs} \equiv \epsilon_{krs}$ ].

Using the commutator relations

$$[J_X, d] = 0, \quad [J_X, \hat{*}] = 0, \quad [J_X, dr] = 0, \quad (D1)$$

where  $X$  is an infinitesimal rotation, and hence a Killing field for  $\mathbf{g}$ , it is not difficult to see that

$$Y dr, \quad dY, \quad -\hat{*}dY$$

form a basis of spherical harmonic *one-forms* with total angular momentum  $\ell$ , where  $Y \equiv Y^{\ell m}$  are the standard *scalar* spherical harmonics. The dual basis,  $\underline{C}_1 \equiv Y \underline{e}_r$ ,  $\underline{C}_2 \equiv \hat{g}^{AB} \hat{\nabla}_B Y \underline{e}_A$ ,  $\underline{C}_3 \equiv \hat{\eta}^{AB} \hat{\nabla}_B Y \underline{e}_A$  is a linear combination of the standard *vector* harmonics (see, e.g. [24]). (Here and in the following,  $\underline{e}_k$  denote the standard basis fields of  $\mathbb{R}^3$ ,  $\underline{e}_r$  is the radial unit vector, and  $\underline{e}_A$  is a basis of  $S^2$  with dual basis  $\hat{\theta}^A$ . The antisymmetric tensor  $\hat{\eta}_{AB}$  is defined by  $\hat{*}\hat{\theta}^A = \hat{\eta}^A_B \hat{\theta}^B$ .) Since the operator  $d$  is parity preserving, while the operator  $\hat{*}$  is parity reversing,  $\underline{C}_1$  and  $\underline{C}_2$  have *even* parity, while  $\underline{C}_3 = \hat{g}^{AB} S_A \underline{e}_B$  has *odd* parity. Here  $S_A \equiv \hat{\eta}_{AB} \hat{\nabla}^B Y$  denote the transverse spherical vector harmonics,  $\hat{g}^{AB} \hat{\nabla}_B S_A = 0$ .

In order to construct su(2)-valued spherical harmonics, we use the isometry  $\underline{e}_k \leftrightarrow \tau_k$  to identify  $\mathbb{R}^3$  with su(2), where the standard inner product on  $\mathbb{R}^3$  corresponds to the normalized inner product  $\text{Tr} \equiv -2\text{trace}$  on su(2). Vector-valued tensors are identified with su(2)-valued tensors, and the operator  $d$  is defined by the exterior derivative  $\mathcal{D}$  for vector-valued forms  $\underline{\alpha} = \alpha^i \underline{v}_i$ :  $d\underline{\alpha} = \underline{v}_i \mathcal{D}\alpha^i = \underline{v}_i (d\alpha^i + \omega^i_j \wedge \alpha^j)$ , where  $\omega^i_j$  is the Riemannian connection with respect to the standard metric on  $\mathbb{R}^3$ . (With respect to the standard basis,  $\underline{v}_i = \underline{e}_i$ , one has  $\omega^i_j = 0$ , and thus  $\mathcal{D}\underline{\alpha} = d\underline{\alpha}$ , whereas, with respect to the basis vectors  $\underline{e}_r$  and  $\underline{e}_A$ , one finds  $\omega_r^A = \hat{\theta}^A$ ,  $\omega_A^r = -\hat{g}_{AB} \hat{\theta}^B$ , and  $\omega_A^B = \hat{\omega}_A^B$ .) The basis of su(2)-valued spherical harmonics becomes

$$X_1 = Y \tau_r, \quad X_2 = \hat{g}^{AB} \tau_A \hat{\nabla}_B Y, \quad X_3 = \hat{\eta}^{AB} \tau_A \hat{\nabla}_B Y, \quad (D2)$$

that is,  $X_j \equiv \underline{C}_j \cdot \underline{\tau}$ . Since the parity operator does not act on the inner index,  $X_1$  and  $X_2$  have *odd* parity, while  $X_3$  has *even* parity.

A basis of su(2)-valued spherical harmonic *one-forms* is now obtained by the same procedure as above: Using the commutator relations (D1), with  $d$  generalized as above, one obtains the nine basis vectors  $dX_k$ ,  $\hat{*}dX_k$ ,  $X_k dr$ . The decomposition  $d\alpha = \tau_i \mathcal{D}\alpha^i = \hat{\nabla}\alpha - \tau_r \hat{g}_{AB} \hat{\theta}^B \wedge \alpha^A$  of the total exterior derivative of a vector valued form  $\alpha$  tangential to  $S^2$  now yields the identities

$$\begin{aligned} dX_1 &= Y d\tau_r + \tau_r dY, \\ dX_2 &= \hat{\nabla} X_2 - \tau_r dY, \\ dX_3 &= \hat{\nabla} X_3 + \tau_r \hat{*}dY. \end{aligned} \quad (D3)$$

Furthermore, one has

$$-\hat{*}\hat{\nabla}X_3 + \hat{\nabla}X_2 + \ell(\ell+1)Yd\tau_r = 0. \quad (D4)$$

By virtue of these identities one may also use the one-forms  $Yd\tau_r$ ,  $\tau_r dY$ ,  $\hat{\nabla}X_2$  instead of  $dX_1$ ,  $dX_2$ ,  $\hat{*}dX_3$ , or the one-forms  $Y\hat{*}d\tau_r$ ,  $\tau_r\hat{*}dY$ ,  $\hat{*}\hat{\nabla}X_2$  instead of  $\hat{*}dX_1$ ,  $\hat{*}dX_2$ ,  $dX_3$ . In fact, the new sets turn out to be more convenient in order to derive the perturbation equations.

In conclusion, the  $\mathfrak{su}(2)$ -valued spherical harmonic basis one-forms with odd parity are

$$X_1 dr, \quad X_2 dr, \quad Yd\tau_r, \quad \tau_r dY, \quad \hat{\nabla}X_2, \quad (D5)$$

while the even parity basis one-forms are

$$X_3 dr, \quad Y\hat{*}d\tau_r, \quad \tau_r\hat{*}dY, \quad \hat{*}\hat{\nabla}X_2. \quad (D6)$$

This is, however, only true for  $\ell > 1$ . For  $\ell = 1$  and  $\ell = 0$  the above fields are not linearly independent. For  $\ell = 1$  the dimensions of both the odd and the even parity sectors are reduced by one, since  $\hat{\nabla}_A \hat{\nabla}_B Y^{(\ell=1)} = -\hat{g}_{AB} Y^{(\ell=1)}$  implies  $\hat{\nabla}X_2 = \hat{g}^{BC} \hat{\nabla}_A \hat{\nabla}_B Y \tau_C \hat{\theta}^A = -Y \tau_A \hat{\theta}^A = -Y d\tau_r$ . For  $\ell = 0$ ,  $Y$  is constant, and hence  $X_2$ ,  $X_3$ , and  $dY$  vanish. Specially, in the even parity case only  $\hat{*}d\tau_r$  survives, which yields the spherically symmetric magnetic Witten ansatz for the gauge potential.

It is also worthwhile noticing that the odd-parity expansion (3) of the metric perturbations can be obtained by “lowering the inner index” and symmetrizing the one-forms (D6):

$$\begin{aligned} X_3 dr &= \hat{g}^{AB} S_A \tau_B dr \rightarrow \delta g = S_A (dr \otimes \hat{\theta}^A + \hat{\theta}^A \otimes dr), \\ Y\hat{*}d\tau_r &= Y \tau_A \hat{\eta}_B^A \hat{\theta}^B \rightarrow \delta g = 0, \\ \tau_r \hat{*}dY &= \tau_r S_A \hat{\theta}^A \rightarrow \delta g = S_A (dr \otimes \hat{\theta}^A + \hat{\theta}^A \otimes dr), \\ \hat{*}\hat{\nabla}X_2 &= \hat{g}^{BC} \tau_B \hat{\nabla}_C S_A \hat{\theta}^A \rightarrow \delta g = \hat{\nabla}_{\{A} S_{B\}} \hat{\theta}^A \otimes \hat{\theta}^B. \end{aligned}$$

In a similar manner the even-parity metric expansion can be obtained from the (odd-parity) one-forms (D5).

## APPENDIX E: INVARIANT YANG-MILLS PERTURBATIONS

In this Appendix we construct the gauge- and coordinate-invariant amplitudes parameterizing the perturbations of the YM potential  $\delta A$ . Starting with Eqs. (27), (28) and (29), our aim is to show that the physical perturbations for  $\ell > 1$ ,  $\ell = 1$  and  $\ell = 0$  are given by the expressions (30), (31) and (33), respectively.

Under YM gauge transformations one has

$$\delta A \rightarrow \delta A + D\chi,$$

where  $D$  is the gauge covariant derivative with respect to the background connection (18), and  $\chi$  denotes the  $\mathfrak{su}(2)$ -valued scalar field parameterizing the gauge freedom. For odd parity perturbations  $\chi$  is given in terms of two functions on  $\tilde{M}$ ,

$$\chi = f_1 X_1 + f_2 X_2,$$

where  $X_1$  and  $X_2$  are the odd-parity scalar isospin harmonics defined in Eq. (D2).

Now using the identities (D3) and (D4) one finds  $DX_1 = \tau_r dY + wY d\tau_r$ ,  $DX_2 = \hat{\nabla}X_2 - w\tau_r dY$ , the amplitudes defined in Eqs. (27) and (28) are found to behave as follows under gauge transformations:

$$\begin{aligned} \alpha &\rightarrow \alpha + df_1 \\ \beta &\rightarrow \beta + df_2 \\ \mu &\rightarrow \mu + f_1 - f_2 w \end{aligned} \quad \text{for } \ell \geq 1, \quad (E1)$$

and

$$\nu \rightarrow \nu + f_1 w - f_2 \quad \text{for } \ell = 1, \quad (E2)$$

$$\begin{aligned} \nu &\rightarrow \nu + f_1 w \\ \sigma &\rightarrow \sigma + f_2 \end{aligned} \quad \text{for } \ell > 1. \quad (E3)$$

For  $\ell = 1$ , one can introduce two gauge-invariant one-forms  $a$  and  $b$ , say,

$$\begin{aligned} a &\equiv \alpha - d\left(\frac{\mu - w\nu}{1 - w^2}\right) \\ b &\equiv \beta + d\left(\frac{\nu - w\mu}{1 - w^2}\right) \end{aligned} \quad \text{for } \ell = 1, \quad (E4)$$

which are well-defined unless the background configuration is the Schwarzschild black hole,  $w = 1$ . The transformation laws (E1) and (E2) imply that there exists a gauge for which the scalars  $\mu$  and  $\nu$  vanish. Moreover, the above definitions show that in this gauge the one-forms  $a$  and  $b$  coincide with the gauge invariant one-forms  $a$  and  $b$ . Since the perturbation equations are gauge-invariant, we may thus parametrize  $\delta A^{(\ell=1)}$  in terms of the two gauge-invariant one-forms  $a$  and  $b$  on  $\tilde{M}$ ,

$$\delta A^{(\ell=1)} = X_1 a + X_2 b. \quad (E5)$$

For  $\ell > 1$ , we may proceed in a similar way and introduce two gauge-invariant one-forms and one gauge-invariant function as follows:

$$\begin{aligned} a &\equiv \alpha - d(\mu + w\sigma) \\ b &\equiv \beta - d\sigma \\ c &\equiv \nu - w(\mu + w\sigma) \end{aligned} \quad \text{for } \ell > 1. \quad (E6)$$

It is again obvious from Eqs. (E1) and (E3) that there exists a gauge for which  $\mu$  and  $\sigma$  vanish, and that the remaining amplitudes  $\alpha$ ,  $\beta$  and  $\nu$  coincide with the gauge-invariant quantities  $a$ ,  $b$  and  $c$  in this gauge. Hence, without loss of generality, we may set

$$\delta A^{(\ell>1)} = X_1 a + X_2 b + c Y d\tau_r, \quad (E7)$$

and consider  $a$ ,  $b$  and  $c$  as gauge-invariant amplitudes.

For  $\ell = 0$ ,  $\delta A$  is parametrized in terms of the one-form  $\alpha$  and the function  $\nu$ , which transform according to  $\alpha \rightarrow \alpha + df_1$  and  $\nu \rightarrow \nu + f_1 w$ , respectively. The amplitudes combine into a gauge-invariant one-form

$$a \equiv \alpha - d\left(\frac{\nu}{w}\right) \quad \text{for } \ell = 0, \quad (\text{E8})$$

where  $\alpha$  coincides with  $a$  in the gauge for which  $\nu$  vanishes. (This gauge does not exist for the RN background, since  $\nu$  is gauge-invariant for  $w(r) \equiv 0$ .) In terms of  $a$  one has

$$\delta A^{(\ell=0)} = \tau_r a. \quad (\text{E9})$$

So far we have parametrized  $\delta A$  in terms of gauge-invariant amplitudes, or, more precisely, in terms of amplitudes which coincide with gauge-invariant amplitudes in a certain gauge. However, these quantities are not yet invariant under infinitesimal coordinate transformations on the background. As the linearized Einstein and YM equations are invariant under these transformations, they will involve only coordinate-invariant combinations of the above amplitudes. In order to find these combinations, it remains to study the behavior of the gauge-invariant amplitudes  $a$ ,  $b$  and  $c$  under the transformation

$$\delta A \rightarrow \delta A + \mathcal{L}_X A,$$

where  $A$  is the background connection given in Eq. (18), and  $\mathcal{L}_X$  denotes the Lie derivative with respect to the infinitesimal vector field  $X^\mu = -fR^{-2}\delta_A^\mu \hat{\gamma}^{AB} \hat{\nabla}_B Y$ , defined in Eq. (4). In terms of the coordinate freedom  $f$ , one finds

$$\mathcal{L}_X A = (1-w) \left[ \frac{f}{R^2} (\tau_r dY + \hat{\nabla} X_2) + X_2 d\left(\frac{f}{R^2}\right) \right].$$

(The most efficient way to establish this is to write  $\mathcal{L}_X = di_X + i_X d$ , and to use  $i_X d\Omega = -R^{-2}f dY$  and  $i_X \hat{*} d\tau_r = R^{-2}f X_2$ .)

The transformation properties of the one-forms  $\alpha$ ,  $\beta$  and the functions  $\mu$ ,  $\nu$  and  $\sigma$  defined in Eq. (27) are now immediately obtained. [For  $\ell = 1$  one has to replace  $\hat{\nabla} X_2$  by  $-Y d\tau_r$  and to use Eq. (28) instead of Eq. (27).] For  $\ell > 1$ , the gauge-invariant quantities (E6) transform as follows under coordinate transformations generated by  $X$ :

$$\begin{aligned} a &\rightarrow a - d\left[\frac{f}{R^2}(1-w^2)\right] \\ b &\rightarrow b + \frac{f}{R^2} dw \\ c &\rightarrow c - \frac{f}{R^2} w(1-w^2) \end{aligned} \quad \text{for } \ell > 1, \quad (\text{E10})$$

while the transformation laws for the quantities (E4) become

$$\begin{aligned} a &\rightarrow a - d\left(\frac{f}{R^2}\right) \\ b &\rightarrow b - w d\left(\frac{f}{R^2}\right) \end{aligned} \quad \text{for } \ell = 1. \quad (\text{E11})$$

(There exist no allowed coordinate transformations in the odd-parity sector if  $\ell = 0$ .) For  $\ell > 1$ , one may eventually use the transformation property (5) of the metric perturbation  $\kappa$ ,  $\kappa \rightarrow \kappa + f$ , to introduce the following gauge and coordinate-invariant amplitudes:

$$\begin{aligned} A &\equiv a + d\left[\frac{\kappa}{R^2}(1-w^2)\right], \quad B \equiv b - \frac{\kappa}{R^2} dw \\ C &\equiv c + \frac{\kappa}{R^2} w(1-w^2), \quad H \equiv h - R^2 d\left(\frac{\kappa}{R^2}\right), \end{aligned} \quad (\text{E12})$$

where we have also recalled the definition (10) of the coordinate-invariant metric perturbation one-form  $H$ . In the ODG ( $\kappa = 0$ ) these gauge- and coordinate-invariant amplitudes coincide with the gauge-invariant amplitudes  $a$ ,  $b$ ,  $c$ , and  $h$ , which reduce to the original amplitudes  $\alpha$ ,  $\beta$ ,  $\gamma$ , and  $h$  in the ODSG ( $\kappa = \mu = \sigma = 0$ ).

For  $\ell = 1$  the gauge- and coordinate-invariant YM amplitudes are obtained by comparing the transformation laws (E11) with the transformation property (5) of the metric perturbation  $h$ ,  $h \rightarrow h + R^2 d(R^{-2}f)$ . This yields the invariant quantities  $\bar{a}$  and  $\bar{b}$ , defined by

$$\bar{a} \equiv a + \frac{h}{R^2}, \quad \bar{b} \equiv b + w \frac{h}{R^2}. \quad (\text{E13})$$

## APPENDIX F: LINEARIZED FLUX INTEGRALS

The Komar expressions for the local electric and magnetic charges, the local mass and the local angular momentum of a stationary spacetime are given by the following flux integrals over a sphere with radius  $R$ :

$$\begin{aligned} Q_e(R) &= \frac{1}{4\pi} \int_{S_R} *F, \quad Q_m(R) = \frac{1}{4\pi} \int_{S_R} F, \\ M(R) &= -\frac{1}{8\pi G} \int_{S_R} *(dg_{t\mu} \wedge dx^\mu), \\ J(R) &= \frac{1}{16\pi G} \int_{S_R} *(dg_{\varphi\mu} \wedge dx^\mu). \end{aligned}$$

Using the expressions (30) and (31) for the gravitational and the YM perturbations, the linearized flux integrals are found to be

$$\begin{aligned} \delta Q_m(R) &= \delta M(R) = 0, \\ \delta Q_e(R) &\sim \delta_{\ell 1} \underline{e}_m \cdot \underline{\tau} R^2 (\tilde{*}F_A + 2\tilde{*}F_B), \\ \delta J(R) &\sim \delta_{\ell 1} \delta_{m 0} R^4 \tilde{*}F_H, \end{aligned} \quad (\text{F1})$$

where  $\underline{e}_0 = (0, 0, 1)$ ,  $\underline{e}_\pm = (\mp 1, i, 0)$ , and  $F_H$ ,  $F_A$  and  $F_B$  are defined in Eq. (52). Here we have also used the orthogonality of the spherical harmonics  $Y^{\ell m}$  and the expansions  $\underline{e}_r \sim Y^{1m} \underline{e}_m$  and  $S_\varphi^{\ell m} \sim \hat{g}(dY^{\ell m}, dY^{10})$ .

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